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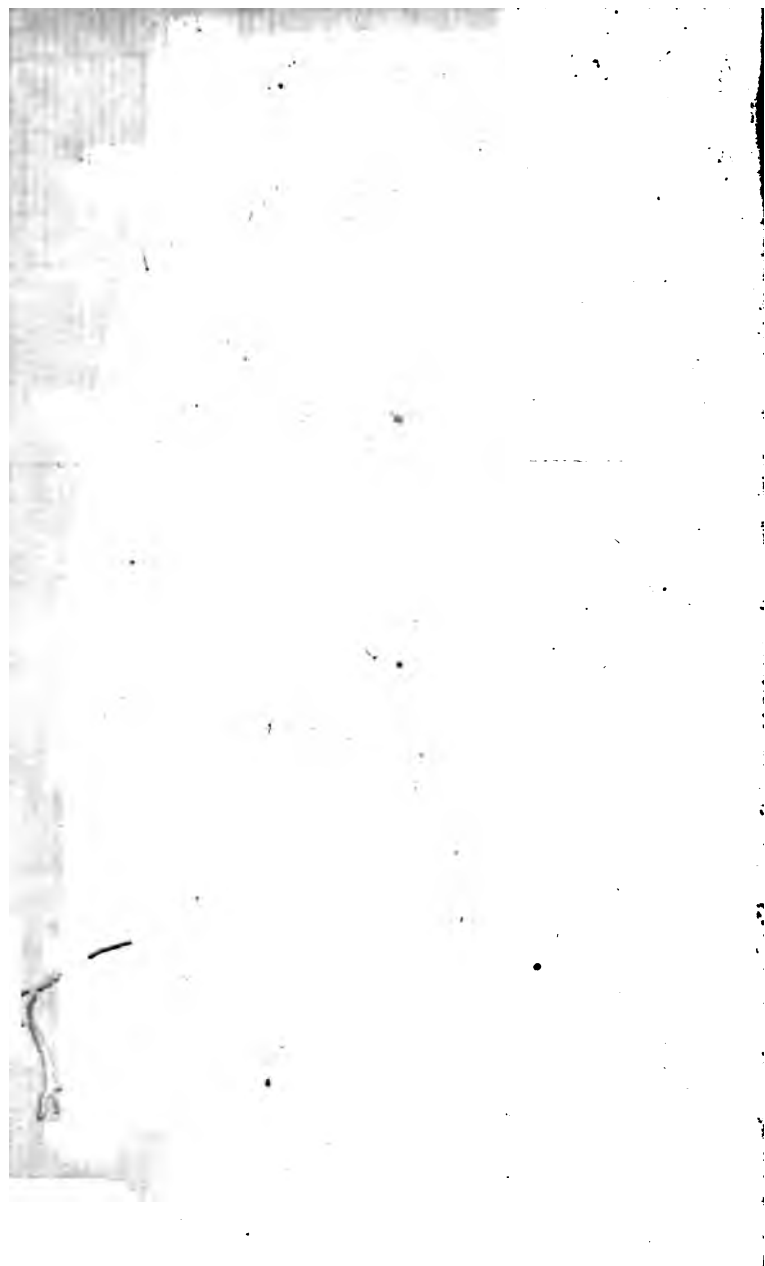
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Friends' School in Kendal.

Began the 27th of the 4th mo. 1772:

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THE
MATHEMATICAL
REPOSITORY.

VOL. II.

CONTAINING
Algebraical Solutions

OF

A great Number of PROBLEMS,

In several Branches of the

MATHEMATICS.

- I. Indetermined Questions, solved generally, by an elegant Method communicated by Mr. *De Moivre*.
- II. Many curious Questions relating to Chances and Lotteries.
- III. A great Number of Questions concerning Annuities for Lives, and their Reversions; wherein that Doctrine is illustrated in a Multitude of interesting Cases, with numeral Examples, and Rules in Words at length, for those who are unacquainted with the Elements of these Sciences, &c.

By *JAMES DODSON*,
Accountant, and Teacher of the MATHEMATICS.

LONDON,
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Hist. of science

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T O

David Papillon, Esq; F.R.S.

One of the Honourable Com-
missioners of the Excise.

S I R,

ALthough the valuation of annuities on
lives, the subject principally treated of
in this book, is of great importance in most
parts of the known world, and particularly in
the *British* empire; yet it is a branch of ma-
thematical learning, which has been but lately
cultivated, and probably not yet brought to
that perfection which it is capable of.

I have ventured to publish the result of my
endeavours to facilitate this kind of interesting
and difficult computations, because I conceiv'd
that I had made some improvement therein;
and I have been encouraged to address it to
you, because you are known to be a gentleman,
who is, not only, well acquainted with the

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mathematical sciences, but also, an encourager of those who attempt to promote useful learning.

Should you honour this work with a perusal, I hope, SIR, you will excuse, or at least pass a favourable censure upon, the imperfections which your great knowledge, in those things, may discover: These, I hope, will be few, because I have endeavoured (to the utmost of my ability) to present you with a treatise, that might be worthy of your acceptance and approbation.

I am,

S I R,

Your most obliged,

And most humble Servant,

JAMES DODSON.

THE PREFACE.

THE general method of solving indetermined questions, which is introduced at the beginning of this volume, was composed by the celebrated mathematician, whose name it bears, some years ago, and was then conceived in the form of a letter, directed to William Jones, Esq; with a design that the same should have been printed in the philosophical transactions, in that form; but Mr. Jones's death having prevented that, it was generously communicated to the author, to be inserted in this work, in such a form as would be conformable thereto.

This communication determined the author to reassume the subject of indetermined questions, which, though it bears a place in the first volume, is not there handled, either so generally, or so elegantly, as here.

Mr. De Moivre proposed to shew the manner of solving those questions, in which there are three unknown quantities and but one equation; and in order to render that the more intelligible, he introduced one example, in which there were but two unknown quan-

ties, to which the author hath added a few more, to make that method of solution familiar to the reader.

Mr. De Moivre's solution of equations, containing three unknown quantities, is truly elegant, and equal to his other performances; and, as the author found that those methods might be extended to equations in which there are four unknown quantities, he has endeavoured to shew how it may be done; but lest any inelegance or inaccuracy of his should, for want of a proper distinction, be imputed to Mr. De Moivre, he has remarked the place where that gentleman's performance ends.

In the second part of the first volume of this work series of different kinds are treated of in a general manner, the application of many of which, to real use, is not therein contained; it was therefore thought convenient to resume that subject also, and, for the sake of students in algebra, to shew in what manner they are actually conducive to the facilitating the computations of many questions, which frequently occur in the transactions of mankind.

And here it was impossible to pass over their usefulness in a science, which (although at all times necessary to be understood) was altogether unknown to the antient mathematicians, and has been but very lately brought to any degree of perfection; the intelligent reader will easily perceive, that the doctrine of chances is here meant; upon which subject there are hitherto but few writers in any language, and Mr. De Moivre, the author of the first system thereof in English, is yet alive.

It

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It will therefore be very easy for the reader to see the reason, why there is not such a pompous list of authors quoted in this preface, as in that to the former volume; for when Huygens, Monmort, and Bernoulli; De Moivre, and Simpson, have been named, the other writers, on the subject, must be searched for among the transactions of the several royal societies, and other miscellaneous and periodical works.

The author found himself obliged to introduce the solutions of several questions, relating to this science, which do not depend on series, in order to enable the learner to understand the reasons of those solutions which do; and in these, because they contain as it were the principles of the science, he has been obliged to follow one, or more, of the above-named authors strictly; but in the solutions of questions, which are not so fundamental, he has frequently aimed at an improvement, and hopes with some success.

But of all the various kinds of problems relating to chance, there are none so interesting, to the inhabitants of these kingdoms, as those relating to annuities for lives, and the reversions of them; which will be evident when we consider the great property vested in them.

The present possessors of entailed estates, are in the common law justly called tenants for life; marriage settlements generally convey the reversion of a considerable part of the bridegroom's estate, to the bride,

for her natural life after his decease; to which two things all the freehold estates in these kingdoms are liable; and if to these be added, the great number of copyholds determinable on lives; the great quantities of church, college, and other lands leased on lives; and the estates possessed by ecclesiastical persons of all degrees, we shall find, that the values of the possessions, and the reversions, of much the greatest part of the real estates, in these kingdoms, will, one way or other, depend on the values of lives.

Likewise the incomes annexed to all places, civil and military, all pensions, and most charitable donations, are annuities for life; the interests or dividends of many personalities in the stocks have been, by the wills of their possessors, rendered of the same kind; besides which there are some annuities on lives which have been granted by the government, and have parliamentary security for their payment; and others that have been granted by parishes, and other communities, in consequence of acts of parliament made for that purpose.

To the solution of questions of this sort, therefore, the author hath, in this volume, applied the summation of those kind of series which the reader will find, in questions 185 to 193, part II. vol. I. having first, in questions 15 to 20 hereof, for the sake of perspicuity, represented and summed such particular series as could be applied to this purpose, in a manner somewhat different from those above quoted.

These

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Those preparatory questions are placed directly after those of the indetermined kind, because the chain of those relating to chance should be uninterrupted; and are followed by questions 21 and 22, which (till their application, to the approximation of the values of joint lives, appears in questions 64 and 69) seem to have no relation to the rest of the work; and may, therefore, if the reader chooses, remain unread till wanted.

The solutions of the following questions, to the 29th, exhibit the first principles of the doctrine of chances; and though the words of each of these questions state only a particular case, they are of general application.

The subsequent questions are classed, and contain the following probabilities, viz.

The 30th and 31st, those of the happening of one event, or more, in 2, 3, or m trials;

The 32d, 33d, and 34th, those of one event and no more in 2, 3, 4, or m trials;

The 35th and 36th, those of two events, or more, in 3, 4, or m trials;

And, the 37th, 38th, and 39th, those of two events, and no more, in 3, 4, 5, or m trials.

The questions, from 40 to 44, both inclusive, apply this doctrine to the play at Backgammon.

The solutions from thence, to question 48, are introduced to shew, that all questions, that can possibly be asked, concerning the happening, or failing of any number of events, in 2, 3, 4, or m trials, may be answered by one, or more, of the terms, of the 2d, 3d, 4th, or mth power, of that binomial, whose root is the sum of the chances, for the happening and failing of one such event.

The questions, numbered 49 to 54, relate to lotteries, and are introduced for the sake of the general solution, given in question 55, for determining how many tickets ought to be purchased, to procure an equal chance for the obtaining, at least, of 1, 2, 3, 4, or p, prizes.

*Previous to our speaking of the questions relating to annuities on lives, and their reversions, it may be convenient to observe, that the antient ways of determining these values depend upon different customs, which seem to have been established, in the places where they are used, merely for want of good methods of calculation, which customs are still in use in some places; but as the advantages that will attend the determination of these things, by computation, preferably to these customs, are obvious; it may seem strange, that (notwithstanding many of these tenures have subsisted from the very origin of private property in these kingdoms, yet) we do not meet with so much as an attempt towards computing their values, till that of the late justly celebrated Dr. Halley, by the assistance of the
bills*

bills of mortality at Breslaw in Silesia; which was soon followed by Mr. De Moivre's truly admirable hypothesis, that the decrements of life may be esteemed nearly equal, after a certain age.

It has been the opinion of some authors, that since this hypothesis was originally derived from the Breslaw observations, it cannot be near so well adapted to the inhabitants of these kingdoms, as what has been deriv'd from the bills of mortality of London; but this argument doth not (as the author conceives) appear to be conclusive.

First, because those bills, as hitherto kept, are not well adapted to answer this purpose.

Secondly, because the manner in which the inhabitants of London, and those of most of the country towns and villages, live; their occupations, diet, and diversions; nay, the very air they breathe, are as different, as those of London and Breslaw can possibly be; and consequently, so must the times of their dissolution; all which has been, with a great deal of clearness, evidenced by Mr. Corbyn Morris in a pamphlet, called, Observations on the past growth and present state of London.

Thirdly, because those persons, who suppose that Mr. De Moivre's hypothesis has its foundation, particularly, in the Breslaw observations, are greatly mistaken; for, on the contrary, if the London observations

ations had been then in Mr. De Moivre's hands, he might, as justly, have derived his hypothesis from them; which will appear from his own words, in the preface to his treatise of annuities on lives, compared with the London observations.

“ Two or three years after the publication of my doctrine of chances (*says that excellent mathematician*) I took the subject into consideration; and consulting Dr. Halley's table of observations (*See page 149 of this work*) I found that the decrements of life, for considerable intervals of time, were in arithmetic progression; for instance, out of 646 persons of twelve years of age, there remain 640 after one year, 634 after two years, 628, 622, 616, 610, 604, 598, 592, 586, after 3, 4, 5, 6, 7, 8, 9, 10, years respectively, the common difference of those numbers being 6.

“ Examining afterwards other cases, I found that the decrements of life, for several years, were still in arithmetic progression; which may be observed from the age of 54 to the age of 71, where the difference for 17 years together, is constantly 10.

“ After having thoroughly examined the tables of observations, and discovered that property of the decrements of life, I was inclined to compose a table of the values of annuities on lives, by keeping

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“ keeping close to the table of observations ; which
 “ would have been done with ease, by taking, in
 “ the whole extent of life, several intervals, whe-
 “ ther equal or unequal ; however, before I under-
 “ took the task, I tried what would be the result
 “ of supposing those decrements uniform, from the
 “ age of twelve ; being satisfied that the excesses
 “ arising on one side, would be nearly compen-
 “ sated by the defects on the other ; then compar-
 “ ing my calculation with that of Dr. Halley, I
 “ found the conclusion so very little different, that
 “ I thought it superfluous to join together several
 “ different rules, in order to compose one.”

*Now the same thing, which Mr. De Moivre men-
 tions above, happens in the tables of the London ob-
 servations (see page 157 of this work) viz. out of 510
 persons of twelve years of age, there remain 504 af-
 ter one year, 498 after two years, 492, 486, 480,
 474, 468, 462, after 3, 4 5, 6, 7, and 8 years
 respectively, the common difference being 6 ; and the
 like happens in many other instances ; but the lengths
 of the intervals differ from those in the Breslaw table.*

*Now, since either of those tables of observations
 might have furnished the sagacious Hypothesist with
 the invention thereof ; and since there is no reason to
 doubt, but that the bills of mortality of other places
 will furnish tables having the same general properties
 (although*

(although the lengths of the intervals; and manner of the increase, or decrease, of their differences may not be the same with either of these); therefore it is highly probable, that if the observations, drawn from the bills of mortality of a great number of places, were added together, and a mean table composed therefrom, that the numbers, therein contained, would be, at least for larger intervals than in either of these, truly arithmetical.

And if this should prove so, the hypothesis may be said to be deduced from the bills of mortality of the world; and will be much more generally useful, than any particular table of observations.

However, if the argument for the use of the London observations be admitted, we shall want such tables for every place, wherein a person, whose life is to be valued, may usually reside, in order to be able to calculate it to a sufficient exactness; and these, indeed, it is to be wished, were actually in being; and, whenever such tables can be obtained, a method of calculating therefrom is provided in this work.*

In the mean time, we must have recourse to the hypothesis, for the calculations of such lives as are not resident in London, and a few more great cities.

* See the author's letter upon this subject, page 333, vol. 47, of the philosophical transactions.

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Supposing, therefore, the decrements of life to be equal, the solutions of all questions, that can be asked, concerning annuities on one or more lives, and the reversions of them, are here, for the students ease, answered by the application of the summations of the series in questions 15 to 20; although each of them might have been reduced to a different series, and separately solved, by the methods given in questions 185 to 193, part II. vol. I.

The whole work of these questions is also, for the reader's ease, performed at length, without suppressing any of the intermediate steps; and the results of all those operations, which it was conceived would be of frequent use, are (for the same reason) inserted in words at length, after the example of Mr. De Moivre, and are so contrived, that these rules seldom refer to those going before, in order to discover the necessary method of operation.

Farther, it has not been thought sufficient, barely, to give a solution to these questions, but great pains have been taken (after an expression of the answer has been obtained) to dispose the parts thereof into that order, which appeared to give the easiest numerical process; and, as every step necessary thereto is inserted, the solutions are ('tis true) considerably lengthened; but then, the advantages that the operator will thereby receive, in his computation, and those which the student

dent will obtain, in the management of his algebraic expressions, will ('tis presumed) more than compensate that disadvantage.

An instance of this occurs in question 56; wherein the value of a single life is investigated, by Mr. De Moivre's hypothesis, the result whereof, viz.

$$\frac{n - 1 + p \times r - n}{n \times r - 1^2} \quad (\text{in which } n \text{ signifies the complement}$$

of life, p the present worth of one pound due at the end of that complement, and r the amount of one pound and its interest for one year) differs greatly, in appearance,

$$\text{from } \frac{1 - \frac{r}{n} P}{r - 1}, \text{ the expression given in his treatise of}$$

annuities, n and r having the same signification as before, and P signifying the present worth of an annuity certain, for as many years as are denoted by the complement of life.

Now both of these expressions will give the same result, as will appear by the under-written operations of the example given in his treatise, viz. the value of a life of 50 at five per Cent. where $n = 36$ and $r = 1.05$.

First, according to Mr. De Moivre.

$$\begin{array}{l} \text{Here } P \text{ (taken from the } \left. \begin{array}{l} \text{proper table) is} \\ \text{Multiply by } r, \text{ inverted} \end{array} \right\} \begin{array}{l} 16,547; \\ 30,1, \\ \hline 16,547 \\ 827 \\ \hline 17,374 \end{array} \end{array}$$

Divide

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Divide by $n = 36$) 17,374 (, 4826 quotient.

$$\begin{array}{r} 144 \\ \hline 297 \\ 288 \\ \hline 94 \\ 72 \\ \hline 22 \end{array}$$

The quotient ,4826, being taken from unity, leaves

,5174;
Divide by $(1,05 - 1 =),05$) ,5174
10,35 the value re-
quir'd.

Secondly, according to the results of question 56,
p. (taken from the } ,1731, and $n - 1 = 35$;
proper table) is }

Therefore $n - 1 + p = 35,173$;

Multiply by 1, inverted 50,1,

$$\begin{array}{r} 35,173 \\ 1759 \\ \hline 36,932; \end{array}$$

From which taking } ,932 the dividend,
36 remains }

$1 - 1^2 = (,05 \times ,05 =),0025$;

Multiply by
26.
150

$$\begin{array}{r} 75 \\ \hline ,0900 \text{ the divisor:} \end{array}$$

And

$$,09 \text{) } ,932$$

,10,35 the value requir'd;
which

which second operation is preferable to the former, in two respects.

First, because it can be performed by fewer figures; for in the first 52 figures are absolutely necessary, and in the second but 44.

Secondly, because the tabular number, represented by p , will be much easier obtained (if the proper tables are not at hand) than the tabular number P ; for (since the one is only the present worth of one pound, and the other the present worth of an annuity of one pound for the same time) the latter cannot be found without first computing the former.

The authors who have rejected Mr. De Moivre's hypothesis, and choose to make their computations from particular observations, have given us no methods of computing the values (even of single lives) other, than what are particularly adapted to the two tables of observations above-mentioned, except that general one, of performing a multiplication and division, for every year that the proposed life can possibly continue in being.

*To remedy this defect Mr. De Moivre, in a letter to William Jones, Esq; (published in the philosophical transactions No. 473) has given a method, deduced from fluxions, and requiring the use of an hyperbolic logarithm, for the finding the value of a life from a given table of observations; which method could have no place in this work, because it is professedly written for
the*

the use of those who know nothing more than common algebra. The author, however, has been successful enough to remove the obstacle, and to furnish the reader with a rule, which (considering the great difficulty of the problem) is very easy, requiring only those principles which are necessary to the other questions, and a more frequent use of a table of the present worths of one pound.

In investigating the values of combined lives, the author has adhered more strictly to the hypothesis, than its great author himself has done; who, perceiving, perhaps, that the computations of the values of combined lives would, upon that principle, be more prolix than he could wish, has (in the contrivance of his rules) considered the decrements of life as partly in a constant ratio; whereby he obtains an easier operation, but at the same time confesses it to be but an approximation to the truth; for here, the reader will find all the aforementioned cases of combined lives solved, in a manner strictly conformable to the hypothesis, as well as by Mr. De Moivre's approximation.

*But, because the processes, arising from a strict adherence to the hypothesis, may be esteemed too operose; and Mr. De Moivre's approximation has been objected to, as not sufficiently accurate, the author has (by the assistance of questions 21 and 22) exhibited a new approximation to those values, founded on the properties of arithmetical progressions; the results of
which*

which do not (in all the cases the author has yet had occasion to try) differ from those derived, strictly, from the hypothesis, by $\frac{1}{10}$ of a year's purchase, and require little more work (if any) than those of Mr. De Moivre.

It may be asked, perhaps, why (after having discovered so easy and accurate an approximation to the values of joint lives) the author should still persist in investigating the true solutions to the subsequent questions, which take up a great deal of room, and generally give a numerical process, tedious enough at best; when he might, either have directed those answers to be found by the additions and subtractions of the values of joint lives (as has been done by former authors) or, at most, might have added and subtracted, only, the expressions of the approximations; which he has likewise done, in order to obtain those rules which he has given in words at length?

To this question, it may be properly replied, that it must certainly be very agreeable, to every lover of truth, to have it in his power to be as accurate in his computations as the nature of the subject will admit; that many things open themselves to the reader's view, in the prosecution of those solutions, that will be both entertaining and useful; for, not to mention the generality of the conclusion of each set of questions, it must, no doubt, give him a sensible pleasure to find, that (by adhering strictly to the hypothesis as well as by the approximations.)

mations) the value of the longest of three unequal lives may be found, independently of any previous operation, by a numerical process, at least as short as that necessary to the obtaining the value of three unequal joint lives; which is only one, of the seven questions, that (according to the former authors) must be solved, previous to this; and it is reasonable to suppose, that he will, from the experience of this, be armed with patience and resolution enough, in any similar case, to go through a long operation in hope of a like result; That to a reader of small experience (for whom in particular this work is calculated) these solutions will be useful exercises of the theory of algebra, and render him ready at literal computation: and lastly, that, after all, the resulting numerical operations are not so very operose, as to deter a person, who is used to figures, from computing the answers, truly, whenever the largeness of the property in question, or the accuracy of the data, shall render such calculations necessary, or certain.

The reader will therefore find, in the remaining part of the work, the values of annuities on two unequal joint lives, on 2, 3, 4, or ∞ , equal joint lives; on three unequal joint lives; and, on three joint lives, whereof two are equal, and the third either of a greater or lesser age than those: the values of annuities, to continue during the longest of any such lives; the values of the reversions of sums certain, or estates, after single, or any such combinations of lives; the reversi-

ons

ons of an annuity, on any life, after one life; after two equal, or unequal joint lives; or after the longest of two such lives; also of two such lives, after one: All of them solved, strictly, from the hypothesis, and also by the new approximation; in such a manner, as will enable the reader to calculate the value of any such life, or reversion, without being previously obliged to refer to any of the foregoing solutions.

The author has, indeed, investigated the value of annuities, and reversions, on unequal lives, and their approximations, no farther, than where three such are concerned; as supposing those will be sufficient to answer all the purposes that will be commonly required; but then, when the lives are equal, the computation may be extended to any number of lives; and the numerical operations resulting therefrom, adhering strictly to the hypothesis, are as easy as can be expected, in the solutions of questions of such great difficulty; and, perhaps, cannot be shortened by any approximation.

*Since the work of computing the values of combined lives, by the tables of observations, is exceedingly laborious; as appears by quest. 68, in which the value of two joint lives is computed from the London tables; a method of approximating, very nearly, to them, is, therefore, deduced from question 104; by which the complement of life may be found, which, upon the supposition of equal decrements, will have the same probability of attaining the extremity of old age, as
any*

any given life has, according to that table of observations by which its value might be calculated.

Whence, if the complements of any lives (so found) be substituted for the differences between the given ages and 86, in any of the solutions before given, the result will be nearly equal to the answer, which would arise, by strictly computing by such tables of observations: To this question is annexed a table of such complements, adapted to the London observations, and some examples worked thereby.

The questions following the 104th, relate to the values of the expectations of lives; for finding which Mr. De Moivre has given rules, without their demonstration, having only informed the reader, that he found the value of the expectation of a single life by a calculation deduced from the method of fluxions: but here, the value of the expectation of a single life is found, in a manner similar to that of an annuity thereon; and the values of the expectations of combined lives are deduced therefrom, in the same manner as the approximations to the values of annuities on such lives: and the numerical process, for finding the expectation of the longest of any number of lives, is (as before) shorter, than that, for finding the expectation of any number of joint lives.

The following tables are inserted in this work, viz.

A table of the present worths of one pound, due at the end of any number of years, less than 101.

A table

A table of the present values of annuities on single lives, upon the supposition of equal decrements.

A table of the multiples of the sixth part of the present worth of one pound, due at the end of one year; which is very useful in the above mentioned new methods of approximating to the values of combined lives, and their reversionions.

All which tables are computed at the several rates of 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, and 6, per Cent.

As the design of this second volume is the same with that of the first, viz. the rendering of the more difficult parts of calculation easy, and familiar, to learners; and, as it has been endeavoured, in the execution thereof, to render it not altogether unworthy the perusal of more experienced readers; the author hopes for the same favourable reception, which the first volume has met with.

BELL-DOCK, WAPPING,

May 21, 1753.

THE MATHEMATICAL REPOSITORY.

*The SOLUTION of indetermined Questions
in Affirmative Integers, communicated by Mr.
ABRAHAM DE MOIVRE, Fellow of th.
Royal Societies of London and Berlin.*

QUESTION I.

IT is required to find two affirmative integers; the first of which being multiplied by 35, and the second by 43, the sum of those products may be 4000?

Questions of this kind have been solved by several Mathematicians, this being inserted here, only, as an introduction to those that are more difficult.

SOLUTION.

By the question, $35x + 43y = 4000$,

Or $35x = 4000 - 43y$;

Th. $x = \frac{4000 - 43y}{35}$.

Which fractional expression is to be an integer, by the nature of the question.

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B

Now

Now it is evident that, if from any integer, a lesser integer be taken, the remainder will be an integer: therefore, in order to have a remainder in the simplest terms, let the greatest integer possible, be taken from the above expression.

Now $\frac{4000 - 43y}{35} = 114 - y + \frac{10 - 8y}{35}$; as will appear by actually dividing $4000 - 43y$ by 35.

If therefore $114 - y$ (the greatest integer possible) be taken from the above expression, the remainder must be an integer (suppose r) Then $r = \frac{10 - 8y}{35}$,

$$\text{Or } 35r = 10 - 8y.$$

$$\text{Or } 8y = 10 - 35r,$$

$$\text{Therefore (dividing as before) } y = \left(\frac{10 - 35r}{8} \right) = 1 - 4r + \frac{2 - 3r}{8}.$$

In which expression $1 - 4r$ is the quotient, or greatest integer,

$$\text{And } \frac{2 - 3r}{8} \text{ the remainder; let } s = \frac{2 - 3r}{8};$$

$$\text{Then } 8s = 2 - 3r,$$

$$\text{Or } 3r = 2 - 8s;$$

$$\text{Therefore (dividing again) } r = \left(\frac{2 - 8s}{3} \right) = -2s + \frac{2 - 2s}{3}.$$

In which expression $-2s$ is the quotient, or greatest integer, and $\frac{2 - 2s}{3}$ remains let $t = \frac{2 - 2s}{3}$;

$$\text{Then } 3t = 2 - 2s,$$

$$\text{Or } 2s = 2 - 3t;$$

$$\text{Therefore (dividing once more) } s = 1 - t - \frac{t}{2};$$

From which last expression it is evident, that $\frac{1}{2}t$ must be an integer; let therefore $p = \frac{1}{2}t$; Or $2p = t$;

$$\text{Then by resuming the former equation } s = \left(\frac{2 - 3 \times 2p}{2} \right) = 1 - 3p;$$

And

And $r = \left(\frac{2-1-3p \times 8}{3} \right) \frac{2-8+24p}{3},$

That is $r = \left(\frac{24p-6}{3} \right) 8p-2;$

Again $y = \left(\frac{10-35 \times 8p-2}{8} \right) \frac{10-280p+70}{8},$

That is $y = \left(\frac{80-280p}{8} \right) 10-35p;$

Lastly $x = \frac{4000-43 \times 10-35p}{35},$

That is $x = \left(\frac{3570+43 \times 35p}{35} \right) 102+43p.$

Having thus got equations, expressing the values of x and y , in the terms of p ; any integer may be assumed for p that will render x and y affirmative; but from the equation ($y = 10 - 35p$) it is evident, that p cannot be an affirmative integer; for, if it be, then y will be negative, which is contrary to the intent of the question; p must therefore be, either 0, or a negative integer.

$$\begin{array}{l} \text{If } p = 0; x = 102 + 0 = 102; \text{ and } y = 10 - 0 = 10. \\ \text{If } p = -1; x = 102 - 43 = 59; \text{ and } y = 10 + 35 = 45. \\ \text{If } p = -2; x = 102 - 86 = 16; \text{ and } y = 10 + 70 = 80. \end{array}$$

It will appear very plain, that p cannot receive any more interpretations than the three before-going; if it be considered, that when $p = -3$, then $x = (102 - 3 \times 43 =) 102 - 129$; that is x will be negative.

The above method will be better understood, if illustrated by a few examples.

QUESTION II.

Required the values of x , and y , in the equation $71x + 17y = 1005$? See *Quest.* 229, *Vol.* 1.

SOLUTION.

Since $71x + 17y = 1005$,

Th. $17y = 1005 - 71x$,

And by division $y = \left(\frac{1005 - 71x}{17} \right) = 59 - 4x + \frac{2 - 3x}{17}$;

Put $x = \frac{2 - 3r}{17}$; Then $3x = 2 - 17r$,

Whence $x = \left(\frac{2 - 17r}{3} \right) = 5r + \frac{2 - 2r}{3}$;

Put $s = \frac{2 - 2r}{3}$; Then $2r = 2 - 3s$,

And $r = \left(\frac{2 - 3s}{2} \right) = 1 - s - \frac{s}{2}$; Th. let $p = \frac{s}{2}$;

Then $s = 2p$;

$$r = \left(\frac{2 - 3 \times 2p}{2} \right) = 1 - 3p;$$

$$x = 17p - 5;$$

And $y = \frac{1005 - 71 \times 17p + 355}{17}$,

Or $y = \left(\frac{1360 - 71 \times 17p}{17} \right) = 80 - 71p$;

Now because $x = 17p - 5$. Th. $p = \frac{5}{17}$;

And because $y = 80 - 71p$, Th. $p = \left(\frac{80}{71} \right) = 1 \frac{9}{71}$;

Therefore the only value of p in integers will be 1.

And then $x = 17 - 5 = 12$; And $y = 80 - 71 = 9$.

Q U E S.

QUESTION III.

How many ways can 100 pounds be paid by guineas at 21, and pistoles at 17 shillings each? See *Quest. 225, Vol. 1.*

SOLUTION.

By the quest. $21x + 17y = 2000$,

Or $17y = 2000 - 21x$;

Therefore by division $y = \left(\frac{2000 - 21x}{17} \right) = 117 - x + \frac{11 - 4x}{17}$;

Let $r = \frac{11 - 4x}{17}$; Then $4x = 11 - 17r$;

Therefore $x = \left(\frac{11 - 17r}{4} \right) = 2 - 4r + \frac{3 - r}{4}$;

Let $s = \frac{3 - r}{4}$; Then $4s = 3 - r$;

Therefore $r = 3 - 4s$;

Now $x = \left(\frac{11 - 17 \times 3 - 4s}{4} \right) = \frac{11 - 51 + 68s}{4}$;

Or $x = \left(\frac{68s - 40}{4} \right) = 17s - 10$;

Lastly $y = \frac{2000 - 21 \times 17s + 210}{17}$;

Or $y = \left(\frac{2210 - 21 \times 17s}{17} \right) = 130 - 21s$.

Now because $x = 17s - 10$, Th. $s = \frac{10}{17}$;

And because $y = 130 - 21s$, Th. $s = \left(\frac{130}{21} \right) = 6\frac{4}{21}$;

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Therefore $s = 1, 2, 3, 4, 5, 6$.

And then $\begin{cases} x = 7, 24, 41, 58, 75, 92. \\ y = 109, 88, 67, 46, 25, 4. \end{cases}$

QUESTION IV.

How many values can x and y have, in affirmative integers, when the given equation is $5x + 8y = 1989$?

SOLUTION.

Since $5x + 8y = 1989$,

Th. $5x = 1989 - 8y$,

And $x = \left(\frac{1989 - 8y}{5} \right) = 397 - y + \frac{4 - 3y}{5}$;

Let $r = \frac{4 - 3y}{5}$, Then $3y = 4 - 5r$,

Therefore $y = \left(\frac{4 - 5r}{3} \right) = 1 - r + \frac{1 - 2r}{3}$;

Let $s = \frac{1 - 2r}{3}$, Then $2r = 1 - 3s$,

Therefore $r = \left(\frac{1 - 3s}{2} \right) = \frac{1 - s}{2}$;

Let $t = \frac{1 - s}{2}$, Then $s = 1 - 2t$;

Therefore $s = 1 - 2t$;

And $r = \left(\frac{1 - 3 \times 1 - 2t}{2} = \frac{1 - 3 + 6t}{2} \right) = 3t - 1$;

Also $y = \left(\frac{4 - 15t + 5}{3} \right) = 3 - 5t$;

Lastly $x = \left(\frac{1989 - 24 + 40t}{5} \right) = 393 + 8t$.

Now

Now since $y = 3 - 5t$; it follows, that t is either 0, or a negative integer; and x will become negative, when $8t$ exceeds 393; Therefore the values of t will be $\frac{1}{8} \times 393 + 1 (= 49 + 1) = 50$ the number required.

Hence the first values are $y = 3$, and $x = 393$; and the last will be $y = (3 + 5 \times 49) = 248$; and $x = (393 - 8 \times 49) = 1$.

SCHOLIUM. There are two particular cases, in which the solutions of these questions may be obtained, without the above process.

First, From the solution of quest. 233, part 1, vol. 1. it will appear, That, if the absolute number can be measured by the sum of the coefficients of the two unknown quantities; then those unknown quantities may be severally equal to the number, by which the sum of those coefficients doth measure the said absolute number, and the other answers (if any) may be readily found therefrom.

For example, if $3x + 5y = (19 \times 8) = 152$;
Then $x = y = 19$;

Th. $\begin{cases} x = 4, 9, 14, 19, 24, 29, 34, 39, 44, 49. \\ y = 28, 25, 22, 19, 16, 13, 10, 7, 4, 1. \end{cases}$

Secondly, From the solutions of questions 221 and 223, part 1, vol. 1, it will appear, that, if the absolute number can be measured by either of the coefficients of the indetermined numbers; then the indetermined number, that has the measuring coefficient, may be equal to the difference between the number by which the coefficient measures the absolute number, and the coefficient of the other indetermined number; and that other indetermined number will, at the same time, be equal to the measuring coefficient: whence the other answers are readily found.

EXAMPLE 1. If $3x + 5y = 51$;

Then $5y = (51 - 3x) = 17 - x \times 3$;

Whence by quest. 221. $y = 3, 6, 9, 12, \&c.$

And $17 - x = 5, 10, 15, 20, \&c.$

Th. $(17 - 5)x = 12, 7, 2.$

EXAMPLE 2. If $3x + 5y = 85$.

Then $3x = 85 - 5y = 17 - y \times 5$

Whence by quest. 221. $x = 5, 10, 15, 20, \&c.$

And $17 - y = 3, 6, 9, 12, \&c.$

Th. $y = 14, 11, 8, 5, \&c.$

COROL. I.

Hence, if both the coefficients measure the absolute number, then the least value of each indetermined quantity will be the coefficient of the other.

EXAMPLE, If $3x + 4y = 120$;

Then the least value of x will be 4, and the least value of y will be 3;

And $x = 4, 8, 12, 16, 20, 24, 28, 32, 36.$

$y = 27, 24, 21, 18, 15, 12, 9, 6, 3.$

COROL. II.

If both the coefficients, and also their sum, will measure the absolute number; then (besides the property exhibited in the first corollary) the number of answers will be equal to the sum of the two coefficients, less one.

EXAMPLE I. If $3x + 5y = 120$;

Then, the least value of x will be 5; the least value of y will be 3; and the number of answers will be $(3+5-1=) 7$.

For $x = 5, 10, 15, 20, 25, 30, 35.$

And $y = 21, 18, 15, 12, 9, 6, 3.$

EXAMPLE II. If $4x + 7y = 308$;

Then $x = 7, 14, 21, 28, 35, 42, 49, 56, 63, 70.$

And $y = 40, 36, 32, 28, 24, 20, 16, 12, 8, 4.$

Where the number of answers is $(4+7-1=) 10$.

But

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But even where it will be convenient to use the first exhibited process, some cases will be much easier than others.

For instance, if the greater coefficient, and the absolute number (being both divided by the lesser coefficient) leave the same remainder, (as in the following question), the process will be much shorter than in some of the former cases.

QUESTION V.

How many different solutions, in affirmative integers, can be given to the equation $3x + 5y = 173$?

SOLUTION.

Here $x = \left(\frac{173 - 5y}{3} \right) 57 - y + \frac{2 - 2y}{3}$.

Let $r = \frac{2 - 2y}{3}$; Then $3r = 2 - 2y$;

Th. $y = \left(\frac{2 - 3r}{2} \right) 1 - r - \frac{r}{2}$; If $\frac{r}{2} = s$;
 $r = 2s$;

$y = \left(\frac{2 - 6s}{2} \right) 1 - 3s$;

And $x = \left(\frac{173 - 5 \times 1 - 3s}{3} \right) 56 + 5s$;

Where s must be 0, or a negative number; Therefore the signs being changed,

$y = 1 + 3s$; Th. $s = -\frac{r}{3}$;

$x = 56 - 5s$; $s = \left(\frac{56}{5} \right) 11 \frac{1}{5}$;

And the number of answers will be $1 + 11 = 12$.

B. 5.

QUES.

QUESTION VI.

It is required to find three affirmative integers, such, that the first being multiplied by 3, the second by 5, and the third by 8, the sum of all the products may be 10003?

SOLUTION.

By the quest. $3x + 5y + 8z = 10003$,

Or $3x = 10003 - 5y - 8z$,

Th. (by division) $x = 3334 - y - 2z + \frac{1-2y-2z}{3}$;

Let $r = \frac{1-2y-2z}{3}$; Then $2y = 1-2z-3r$,

Whence $y = \left(\frac{1-2z-3r}{2} \right) - z - r + \frac{1-r}{2}$;

Let $s = \frac{1-r}{2}$; Then $2s = 1-r$,

Therefore $r = 1-2s$;

And $y = \left(\frac{1-2z-3(1-2s)}{2} \right) - z - (1-2s) + \frac{1-2z-3+6s}{2}$;

That is $y = \left(\frac{6s-2z-2}{2} \right) - z - 1 + 3s$;

And $x = \frac{10003 - 8z - 5 \times 3s - z - 1}{3}$;

Or $x = \frac{10008 - 9z - 15s}{3}$;

That is $x = 3336 - z - 5s$.

In which two equations (*viz.* $y = 3s - z - 1$, and $x = 3336 - z - 5s$) s is indeterminated, and z is to be determined with proper limitation.

The

The least interpretation that x can possibly have being 1; the above equations may, by that assumption, shew the limits of s ; for then $3s - x - 1$ will become $3s - 2$, and $s \leq \frac{2}{3}$; And $3336 - x - 5s$ will become $3335 - 5s$, Th. $s \leq \left(\frac{3335}{5} = \right) 667$.

Having thus found the limits of s ; those of x will appear as follow:

	A		B
If $s = 1$;	$x = 3331 - x$;	And $y =$	$2 - x$.
$s = 2$;	$x = 3326 - x$;	$y =$	$5 - x$.
$s = 3$;	$x = 3321 - x$;	$y =$	$8 - x$.
$s = 4$;	$x = 3316 - x$;	$y =$	$11 - x$.
<i>&c.</i>	<i>&c.</i>	<i>&c.</i>	

For the clearer explanation of what follows, let the values of x , collectively taken, be denominated by the letter A, which is placed over them; and the values of y , by the letter B, which is placed over them; Then,

First, Since the absolute numbers in the column A decrease perpetually by the subtraction of 5; the number of terms in that column (after the first) may be found by dividing 3331 (the greatest absolute number) by 5 their common difference: Now the quotient of that division will be 666, and the remainder 1; therefore there will be 666 values of x , in that column, after the first, or 667 in the whole.

Secondly, Since the remainder left after dividing 3331, the greatest absolute number in the column A, by 5, the common difference of those numbers, is 1; it appears, that the last value of x in that column will be $1 - x$; which value of x is useless, for then x will be either 0, or negative; and therefore there will be but 666 useful values of x in that column; and s may be any number less than 667: The three last values of s , x , and y , follow;

If $s = 664$; $x = 16 - z$: And $y = 1991 - z$.

$s = 665$; $x = 11 - z$: $y = 1994 - z$.

$s = 666$; $x = 6 - z$: $y = 1997 - z$.

Thirdly, If we were to determine the limits of z from the first equation of the column A, *viz.* $x = 3331 - z$, we might be apt to conclude, that z might be any number under 3331; and by consequence that it might receive 3330 different interpretations; but considering the corresponding equation in the column B, *viz.* $y = 2 - z$; it appears that the values of z are thereby restrained, and that it must be less than 2: In this case therefore z admits but of one interpretation, *viz.* 1, whence $x = (3331 - 1 =) 3330$; and $y = (2 - 1 =) 1$:

Now $3x = (3 \times 3330 =) 9990$,

$5y = (3 \times 1 =) 5$.

And $8z = (8 \times 1 =) 8$.

10003.

From which operation, we may conclude, that the above values of x , y , and z , are rightly found.

Fourthly, By examining the second values of x and y , (which are $x = 3326 - z$, and $y = 5 - z$) it will appear, that z may be any number under 5, *viz.* either 1, 2, 3, or 4: Again, taking the third value of y , (*viz.* $y = 8 - z$) it is plain, that z may be any number less than 8, and consequently, that it will have 7 values: And hence we may be apt to conclude that the number of interpretations of which z is capable, when s is severally interpreted by 1, 2, 3, 4, 5, &c. will be 1, 4, 7, 10, 13, &c. But,

Fifthly, In the same manner as the values of z , in the column A, are restrained by those in the column B, in the cases before cited; so, after a certain number of terms, will its values, in the column B, be restrained by those in the column A: For, if we were to determine the limits of z from the last value of x and y (*viz.* that wherein $s = 666$) we shall find, in column B, that $y = 1997 - z$; from whence it might be concluded, that

that x may be any number less than 1997, if we did not, in the column A, find, that $x = 6 - x$, and consequently, that x must be less than 6.

Sixthly, From the above examples, it is evident, that the column A will begin to restrain the values of x , in the column B, as soon as the absolute number, in the column A, becomes less than the absolute number in the column B; it remains, therefore, to find when that will happen; in order to which,

Let n = the distance of the terms, from the first, wherein the absolute numbers of the two columns will become equal.

Then, because the absolute numbers of the column A, are a decreasing arithmetical progression, whose greatest term is 3331, and common difference 5; the term, whose distance from the greatest is n , will be $3331 - 5n$.

And because the absolute numbers of the column B are an increasing arithmetical progression, whose least term is 2, and common difference 3; the term, whose distance from the least is n , and will be $2 + 3n$.

$$\text{Hence } 3331 - 5n = 2 + 3n,$$

$$\text{Or } 3331 - 2 = (5n + 3n) = 8n,$$

$$\text{That is } \left(\frac{3329}{8}\right) 416\frac{1}{8} = n.$$

Therefore, for 416 terms, after the first, (that is to say, for 417 terms) the absolute numbers in the column A, exceed those in the column B; but in the 418th term, and all following, the absolute numbers in the column B, exceed those in the column A; for the more clear perception of which, the 416th, 417th, and some following terms, are here set down.

$$i = 416; x = 1256 - x; y = 1247 - x$$

$$i = 417; x = 1251 - x; y = 1250 - x$$

$$i = 418; x = 1246 - x; y = 1253 - x$$

$$i = 419; x = 1241 - x; y = 1256 - x$$

Hence,

A		B	
Now if $p=2$;	$x=4342-3x$;	And $y=5$	$-2x$
$p=3$;	$x=4337-3x$;	$y=8$	$-2x$
$p=4$;	$x=4332-3x$;	$y=11$	$-2x$
$\mathcal{E}c.$	$\mathcal{E}c.$	$\mathcal{E}c.$	

If $p=867$;	$x=17-3x$;	$y=2600-2x$
$p=868$;	$x=12-3x$;	$y=2603-2x$
$p=869$;	$x=7-3x$;	$y=2606-2x$

Where the number of terms is 868.

Now from the Beginning of the column B, we find that

When $p=2$;	Then $x-\left(\frac{1}{2}\right)=2\frac{1}{2}$	Th. x has 2	Values.
$p=3$;	$x-\left(\frac{1}{2}\right)=4$	x 3	
$p=4$;	$x-\left(\frac{1}{2}\right)=5\frac{1}{2}$	x 5	
$p=5$;	$x-\left(\frac{1}{2}\right)=7$	x 6	
$p=6$;	$x-\left(\frac{1}{2}\right)=8\frac{1}{2}$	x 8	
$\mathcal{E}c.$	$\mathcal{E}c.$	$\mathcal{E}c.$	

And from the end of column A, we find that when

$p=869$;	Then $x-\left(\frac{1}{3}\right)=2\frac{1}{4}$	Th. x has 2	Values.
$p=868$;	$x-\left(\frac{1}{3}\right)=4$	x 3	
$p=867$;	$x-\left(\frac{1}{3}\right)=5\frac{2}{3}$	x 5	
$p=866$;	$x-\left(\frac{1}{3}\right)=7\frac{1}{3}$	x 7	
$p=865$;	$x-\left(\frac{1}{3}\right)=9$	x 8	
$p=864$;	$x-\left(\frac{1}{3}\right)=10\frac{2}{3}$	x 10	
$p=863$;	$x-\left(\frac{1}{3}\right)=12\frac{1}{3}$	x 12	
$p=862$;	$x-\left(\frac{1}{3}\right)=14$	x 13	
$\mathcal{E}c.$	$\mathcal{E}c.$	$\mathcal{E}c.$	

But although the two series 2, 3, 5, 6, 8, $\mathcal{E}c.$ and 2, 3, 5, 7, 8, 10, 12, 13, $\mathcal{E}c.$ which contain the number of the different values of x , arising from each term of the said columns, are not themselves arithmetical
pro-

progressions as those resulting from the last question were; yet they may be divided into series that are so: That is to say the series 2, 3, 5, 6, 8, 9, 11, 12, 14, 15, &c. may be divided into the two which follow, *viz.* 2, 5, 8, 11, 14, &c. and 3, 6, 9, 12, 15, &c. both which are arithmetical; And the series 2, 3, 5, 7, 8, 10, 12, 13, 15, 17, 18, 20, &c. may be divided into the following three series which are arithmetical, *viz.* 2, 7, 12, 17, &c. 3, 8, 13, 18, &c. and 5, 10, 15, 20, &c. And therefore may be summed, as the former were, when the greatest term of each series is known.

Now, because the values of x , in the column A, are determined by $\frac{1}{3}$ of the absolute number; and in the column B by $\frac{1}{2}$ thereof; it will follow, that in order to find where the column B ceases to determine the number of the values of x , and the column A begins so to do, it will be proper to make an equation between $\frac{1}{3}$ of the absolute number belonging to the term whose distance from the greatest in the column A is n , and $\frac{1}{2}$ of the absolute number of the corresponding term in the column B; that is, because the absolute numbers differ by 5 and 3,

$$\frac{4342 - 5n}{3} = \frac{5 + 3n}{2}$$

Or $8684 - 10n = 15 + 9n$;

That is $8684 - 15 = (10n + 9n) = 19n$;

Therefore $\frac{8669}{19} = 456\frac{1}{19} = n$.

Whence it follows, that the column B will continue to determine the number of the values of x for 456 terms after the first, that is, for 457 terms; and that the column A will determine the same, for the remaining $(868 - 457) = 411$ terms.

Now, since the two series, depending on the column B, are between them to contain 457 terms, it follows that the first of them will contain $\left(\frac{457}{2} + 1 =\right)$ 229 terms, and the second 228. Whence the greatest term of

Again, since the first and last progression, differ only in the first term, we may write 543 more 1076, 1066, 1056, &c. for them.

The same method may be advantageously pursued, with respect to the two remaining progressions, that have 109 terms; *viz.* if their first-terms be separately considered, then all the progressions will have the same number of terms, *viz.* 108.

Hence the number of answers required may be expressed as follows,

$$\begin{array}{l}
 543 + 108 \text{ terms of } 1076, 1066, 1056, \&c. \\
 + 1084 + 108 \text{ terms of } 1074, 1064, 1054, \&c. \\
 + 541 + 108 \text{ terms of } 536, 531, 526, \&c. \\
 + 108 \text{ terms of } \left. \begin{array}{l} 1080, 1070, 1060, \&c. \\ 539, 534, 529, \&c. \end{array} \right\}
 \end{array}$$

Which may (by addition) be expressed by the following single series:

2168 + 108 terms of 4305, 4265, 4225, &c. of which since the common difference is 40, the last term will be $(4305 - (107 \times 40) = 4280) 25$; And consequently the sum of the series will be $\frac{4305 + 25 \times 108}{2} = 233820 \frac{1}{2}$

to which, if the above reserved sum (2168) be added, the result will be 235988.

But this differs from 298204, the number of answers before found, by 62216; from whence, it may very justly be concluded, that this progress is defective.

Now, in the equation $y = 3s + x - 1$, when x is any number greater than 1, s may be nothing, or a negative number; And how great that negative number may be, depends upon the magnitude of x ; for, by the equation $y = 3s + x - 1$; $s = \frac{x-1}{3}$.

Now the greater limit of x may be obtained from the original equation, $3x + 5y + 19z = 13051$; For $19z = 13051 - 3xz - 5y$, and since the least value of either x or y

x or y is unity, the greatest value of $19x$ will be $(13051-3-5=) 13043$; And therefore x cannot exceed $\left(\frac{13043}{19}=\right) 686\frac{9}{19}$.

Hence if we write 686 for x , in the expression, $s = \frac{x-1}{3}$; it will be $s = \left(\frac{685}{3}=\right) 228\frac{1}{3}$.

Now the equation, $y = 3s + x - 1$, may, when s is 0 or negative, become $y = x - 3s - 1$; And the equation, $x = 4352 - 5s - 8x$, may become $x = 4352 + 5s - 8x$. Then

If	0;	x	1;	x	$x = 4352 - 8x$; $x = 16$ \square 544; has values 542.
$s =$	1;	$y = x - 1$	4;	x	$x = 4357 - 8x$; $x = 48$ \square 548;
$s =$	2;	$y = x - 2$	7;	x	$x = 4362 - 8x$; $x = 78$ \square 545;
$s =$	3;	$y = x - 3$	10;	x	$x = 4367 - 8x$; $x = 108$ \square 545;
$s =$	4;	$y = x - 4$	13;	x	$x = 4372 - 8x$; $x = 138$ \square 546;
$s =$	5;	$y = x - 5$	16;	x	$x = 4377 - 8x$; $x = 168$ \square 547;
$s =$	6;	$y = x - 6$	19;	x	$x = 4382 - 8x$; $x = 198$ \square 547;
$s =$	7;	$y = x - 7$	22;	x	$x = 4387 - 8x$; $x = 228$ \square 548;
$s =$	8;	$y = x - 8$	25;	x	$x = 4392 - 8x$; $x = 258$ \square 549;
$s =$	9;	$y = x - 9$	28;	x	$x = 4397 - 8x$; $x = 288$ \square 549;
$s =$	10;	$y = x - 10$	31;	x	$x = 4402 - 8x$; $x = 318$ \square 550;
$s =$	11;	$y = x - 11$	34;	x	$x = 4407 - 8x$; $x = 348$ \square 550;
$s =$	12;	$y = x - 12$	37;	x	$x = 4412 - 8x$; $x = 378$ \square 551;
$s =$	13;	$y = x - 13$	40;	x	$x = 4417 - 8x$; $x = 408$ \square 552;
$s =$	14;	$y = x - 14$	43;	x	$x = 4422 - 8x$; $x = 438$ \square 552;
$s =$	15;	$y = x - 15$	46;	x	$x = 4427 - 8x$; $x = 468$ \square 553;
$s =$	16;	$y = x - 16$	49;	x	$x = 4432 - 8x$; $x = 498$ \square 554;
$s =$	17;	$y = x - 17$	52;	x	$x = 4437 - 8x$; $x = 528$ \square 554;
$s =$	18;	$y = x - 18$	55;	x	$x = 4442 - 8x$; $x = 558$ \square 554;

Whence

It will be expedient, therefore, as in the last process, to find the greater limit of x ; which (since $9x$ may be

$$= (93256 - 7 - 5) = 93244; \text{ And } \frac{93244}{9} = 10360\frac{4}{9}$$

will be 10360: Whence the equation, $x = 18654 - 7s + x$, may become, $x = (18654 - 7s + 10360) = 29014 - 7s$; And when $x=1$, the least value possible of, $y = 5s - 2x - 2$, will be

$$y = 5s - 4.$$

Now by the equations $\begin{cases} y = 5s - 4; & s \leq 4 \\ x = 29014 - 7s, & s \leq \left(\frac{29014}{7} = 4144\frac{6}{7}\right) \end{cases}$

If

values

$$\begin{array}{llllll} s=1; y=3-2x; x=18647+x; x \leq 1\frac{1}{2}, & \& \leq 0, & \text{has } 1. & & \\ s=2; y=8-2x; x=18640+x; x \leq 4, & \& \leq 0, & \text{has } 3. & & \\ s=3; y=13-2x; x=18633+x; x \leq 6\frac{1}{2}, & \& \leq 0, & \text{has } 6. & & \\ s=4; y=18-2x; x=18626+x; x \leq 9, & \& \leq 0, & \text{has } 8. & & \\ s=5; y=23-2x; x=18619+x; x \leq 11\frac{1}{2}, & \& \leq 0, & \text{has } 11. & & \\ \text{etc.} & \text{etc.} & & \text{etc.} & & \text{etc.} \end{array}$$

Therefore (so long as the absolute number in the value of x continues affirmative) the number of answers, which result from the different assumptions of s , will form the two following arithmetical progressions, *viz.* 1, 6, 11, 16, etc. And 3, 8, 13, etc.

Now 18654, the absolute number in the equation, exhibiting the value of x , being divided by seven, will quote $2664\frac{6}{7}$; And therefore the number 7 can be taken from it but 2664 times, and leave an affirmative remainder; whence in the 2665th value of s it will be negative.

If

If s is	Then	Values.
$2665; y = 13323 - 2x; x = x - 1; x \sqsupset 6661\frac{1}{2}, \& \sqsubset 1,6660.$		
$2666; y = 13328 - 2x; x = x - 8; x \sqsupset 6664, \& \sqsubset 8,6655.$		
$2667; y = 13333 - 2x; x = x - 15; x \sqsupset 6666\frac{1}{2}, \& \sqsubset 15,6651.$		
$2668; y = 13338 - 2x; x = x - 22; x \sqsupset 6669, \& \sqsubset 22,6646.$		
$2669; y = 13343 - 2x; x = x - 29; x \sqsupset 6671\frac{1}{2}, \& \sqsubset 29,6642.$		
$\&c.$	$\&c.$	$\&c. \quad \&c. \quad \&c. \quad \&c.$

Which Series is to be continued to $(4144 - 2664 =)$ 1480 Terms; of which the last follows,

$s = 4144; y = 20718 - 2x; x = x - 10354; x \sqsupset 10359, \sqsubset 10354, \& \text{ has 4 values.}$

Hence when the resulting progressions in each case are combined, we may conclude that 1332 terms of the progression 4, 14, 24, $\&c.$ and 740 terms of the progression 13315, 13297, 13279, $\&c.$ will be the number of answers required $= 8869788 + 4931360 = 13801148.$

The following method of solving this question will prove the above, and will serve as an introduction to what follows, concerning 4 indeterminated quantities.

SOLUTION II.

In the equation $5x + 7y + 9z = 93256,$

When x is 1; Then $5x + 7y = (93256 - 9) 93247$

Th. $x = \left(\frac{93247 - 7y}{5} \right) 18649 - y + \frac{2 - 2y}{5}.$

Let $r = \frac{2 - 2y}{5};$ Then $2y = 2 - 5r,$

And $y = \left(\frac{2 - 5r}{2} \right) 1 - 2r - \frac{r}{2};$

Let $s = \frac{r}{2};$ Then $2s = r,$ And

Since $r = 2s; y = \left(\frac{2 - 10s}{2} \right) 1 - 5s,$

And $x = \left(\frac{93247 - 7 \times \overline{1-5s}}{5} \right) 18648 + 7s$

Where $s \sqsubset \frac{1}{5}$, and $s \sqsubset - \left(\frac{18648}{7} \right) 2664$;

Hence when $s = 0$; $y = 1$; $x = 18648$.

$s = -1$; $y = 6$; $x = 18641$.

Ec.

Ec.

$s = -2662$; $y = 13311$; $x = 14$.

$s = -2663$; $y = 13316$; $x = 7$.

And, in this case, the question has 2664 answers.

When $z=2$; Then $5x+7y = (93256-18) = 93238$.

Th. $x = \left(\frac{93238-7y}{5} \right) 18647 - y + \frac{3-2y}{5}$;

Let $r = \frac{3-2y}{5}$; Then $2y = 3-5r$,

And $y = \left(\frac{3-5r}{2} \right) 1-2r + \frac{1-r}{2}$

Let $s = \frac{1-r}{2}$; Then $2s = 1-r$,

And $r = 1-2s$:

Whence $y = \left(\frac{3-5 \times \overline{1-2s}}{2} \right) 5s-1$;

And $x = \left(\frac{93238-7 \times \overline{5s-1}}{5} \right) 18649-7s$;

Where $s \sqsubset \frac{1}{5}$; And $s \sqsubset \left(\frac{18649}{7} = 2664\frac{1}{7} \right)$;

Hence when $s=1$; $y=5-1=4$; & $x=18649-7=18642$.

And when $s=2664$; $y=13320-1=13319$; & $x=1$.

Th. when $z=2$ the question will have 2664 answers.

When $z=3$; Then $5x+7y = (93256-27) = 93229$.

Therefore $x = \left(\frac{93229-7y}{5} \right) 18645 - y + \frac{4-2y}{5}$;

Let

Let $r = \frac{4-2y}{5}$; Then $2y = 4 - 5r$;

And $y = \left(\frac{4-5r}{2}\right) 2 - 2r = \frac{r}{2}$;

Let $s = \frac{r}{2}$; Then $r = 2s$;

$$y = \left(\frac{4-10s}{2}\right) 2 - 5s;$$

And $x = \left(\frac{93229-7 \times 2-5s}{5}\right) 18643 + 7s$

Where $s \sqsupset \frac{2}{5}$; And $s \sqsubset -2663\frac{2}{5}$.

Hence when $s = 0$; $y = 2$; And $x = 18643$;

$$s = -2663; y = (2 + 13315) = 13317; x = 2.$$

Th. when $x = 3$, the question admits of 2664 answers.

When $x = 4$; Then $5x + 7y = (93256 - 36) = 93220$.

Therefore $x = \left(\frac{93220-7y}{5}\right) 18644 - y - \frac{2y}{5}$;

Let $r = \frac{2y}{5}$; Then $5r = 2y$.

And $y = \left(\frac{5r}{2}\right) 2r + \frac{r}{2}$;

Let $s = \frac{r}{2}$; Then $r = 2s$;

$$y = \left(\frac{10s}{2}\right) 5s;$$

And $x = \left(\frac{93220-7 \times 5s}{5}\right) 18644 - 7s$;

Where $s \sqsubset 0$; And $s \sqsupset \left(\frac{18644}{7}\right) 2663\frac{2}{7}$.

Hence when $s = 1$; $y = 5$; $x = (18644 - 7) = 18637$;

$$s = 2663; y = (2663 \times 5) = 13315; x = 3.$$

Th. when $x = 4$, the question admits of 2663 answers.

When $z=5$; Then $5x+7y=(93256-45)=93211$.

Therefore $x = \left(\frac{93211-7y}{5} \right) = 18642 - y + \frac{1-2y}{5}$;

Let $r = \frac{1-2y}{5}$; Then $2y=1-5r$,

And $y = \left(\frac{1-5r}{2} \right) = -2r + \frac{1-r}{2}$;

Let $s = \frac{1-r}{2}$; Then $2s=1-r$,

And $r = 1-2s$;

$y = \left(\frac{1-5 \times \frac{1-2s}{2}}{2} \right) = 5s-2$;

And $x = \left(\frac{93211-7 \times \frac{5s-2}{2}}{5} \right) = 18645 - 7s$;

Where $s \sqsubset \frac{2}{5}$; and $s \sqsupset \left(\frac{18645}{7} \right) = 2663\frac{4}{7}$.

Hence when $s=1$; $y=(5-2)=3$; $x=18638$;

$s=2663$; $y=(2663 \times 5-2)=13313$; $x=4$.

Th. when $z=5$, the question admits of 2663 answers.

When $z=6$; Then $5x+7y=(93256-54)=93202$.

Therefore $x = \left(\frac{93202-7y}{5} \right) = 18640 - y + \frac{2-2y}{5}$;

Whence (proceeding as when z was equal to 1)

$y=1-5s$;

And $x = \frac{93202-7 \times \frac{1-5s}{2}}{5} = 18639 + 7s$;

Where $s \sqsubset \frac{1}{5}$; And $s \sqsupset -\left(\frac{18639}{7} \right) = 2662\frac{5}{7}$.

And therefore, when $z=6$, s has 2663 values.

When

When $x = 7$; Then $5x + 7y = (93256 - 53) = 93193$.

Th. $x = \left(\frac{93193 - 7y}{5} \right) 18638 - y + \frac{3 - 2y}{5}$;

And (proceeding in the same manner as when $x = 2$)
 $y = 5s - 1$;

And $x = \left(\frac{93193 - 7 \times 5s - 1}{5} \right) 18640 - 7s$;

Where $s \sqsubset \frac{1}{2}$; And $s \sqsupset \left(\frac{18640}{7} \right) 2662\frac{6}{7}$.

When $x = 8$; Then $5x + 7y = (93256 - 72) = 93184$.

Th. $x = \left(\frac{93184 - 7y}{5} \right) 18636 - y + \frac{4 - 2y}{5}$;

And (by continuing the process as when $x = 3$)
 $y = 2 + 5s$;

And $x = \frac{93184 - 7 \times 2 - 5s}{5} = 18634 + 7s$;

Where $s \sqsupset \frac{2}{3}$; And $s \sqsubset - \left(\frac{18634}{7} \right) 2662$.

When $x = 9$; Then $5x + 7y = (93256 - 81) = 93175$.

Th. $x = \left(\frac{93175 - 7y}{5} \right) 18635 - y - \frac{2y}{5}$;

Where $y = 5s$ (by the same steps as when $x = 4$)

And $x = \left(\frac{93175 - 7 \times 5s}{5} \right) 18635 - 7s$;

Where $s \sqsubset 0$; And $s \sqsupset \left(\frac{18635}{7} \right) 2662\frac{1}{7}$.

When $x = 10$; Then $5x + 7y = (93256 - 90) = 93166$.

Th. $x = \left(\frac{93166 - 7y}{5} \right) 18633 - y + \frac{1 - 2y}{5}$;

Where $y = 5s - 2$ (by arguing as when $x=5$)

And $x = \left(\frac{93166 - 5s - 2 \times 7}{5} \right) 18636 - 7s$

Where $s \sqsubset \frac{2}{5}$; and $s \sqsupset \frac{18636}{7} = 2662 \frac{2}{7}$.

From the above operations it is evident, that the same equations, expressing the value of y , recur every 5th process, 5 being the coefficient of x ; and that the corresponding equations, expressing the value of x , differ only by (9) the coefficient of s . That is,

				Values,
If $x=1$:	$x=18648+7s$	$s \sqsubset -2664$	has	2664.
$x=6$:	$x=18639+7s$	$s \sqsubset -2662 \frac{3}{7}$		2663.
$x=11$:	$x=18630+7s$	$s \sqsubset -2661 \frac{2}{7}$		2662.
$x=16$:	$x=18621+7s$	$s \sqsubset -2660 \frac{1}{7}$		2661.
$x=21$:	$x=18612+7s$	$s \sqsubset -2658 \frac{6}{7}$		2659.
$x=26$:	$x=18603+7s$	$s \sqsubset -2657 \frac{5}{7}$		2658.
$x=31$:	$x=18594+7s$	$s \sqsubset -2656 \frac{4}{7}$		2657.
$x=36$:	$x=18585+7s$	$s \sqsubset -2655 \frac{3}{7}$		2655.
$x=41$:	$x=18576+7s$	$s \sqsubset -2653 \frac{2}{7}$		2654.
$x=46$:	$x=18567+7s$	$s \sqsubset -2652 \frac{1}{7}$		2653.
$\xi c.$	$\xi c.$	$\xi c.$		$\xi c.$

And the 2072 term of the series will be,

$x=10356$; $y=1-5s$; $x=9+7s$; $s \sqsubset -1 \frac{2}{7}$, has 2 [values.

This whole series, therefore, is composed of 7 arithmetical progressions, whose common difference is 9, and

the number of terms $\frac{2072}{7} = 296$ each. Also,

If

Values.

If $x=2$;	$x=18649-7s$;	$s \sqsupset 2664\frac{1}{7}$, has	2664.
$x=7$;	$x=18640-7s$;	$s \sqsupset 2662\frac{2}{7}$,	2662.
$x=12$;	$x=18631-7s$;	$s \sqsupset 2661\frac{3}{7}$,	2661.
$x=17$;	$x=18622-7s$;	$s \sqsupset 2660\frac{4}{7}$,	2660.
$x=22$;	$x=18613-7s$;	$s \sqsupset 2659$,	2658.
$x=27$;	$x=18604-7s$;	$s \sqsupset 2657\frac{5}{7}$,	2657.
$x=32$;	$x=18595-7s$;	$s \sqsupset 2656\frac{6}{7}$,	2656.
$x=37$;	$x=18586-7s$;	$s \sqsupset 2655\frac{7}{7}$,	2655.
Et c.	Et c.	Et c.	Et c.

And the 2072 term of the series will be,

$$x=10357; y=5s-1; x=10-7s; s \sqsupset 1\frac{1}{7}, s \text{ has } 1.$$

And therefore this whole series is composed of 7 arithmetical progressions, differing from the former, only in some of the greatest terms.

Values.

If $x=3$;	$x=18643+7s$;	$s \sqsupset -2663\frac{2}{7}$, has	2664.
$x=8$;	$x=18634+7s$;	$s \sqsupset -2662$,	2662.
$x=13$;	$x=18625+7s$;	$s \sqsupset -2660\frac{5}{7}$,	2661.
$x=18$;	$x=18616+7s$;	$s \sqsupset -2659\frac{6}{7}$,	2660.
$x=23$;	$x=18607+7s$;	$s \sqsupset -2668\frac{1}{7}$,	2659.
$x=28$;	$x=18598+7s$;	$s \sqsupset -2656\frac{2}{7}$,	2657.
$x=33$;	$x=18589+7s$;	$s \sqsupset -2655\frac{3}{7}$,	2656.
$x=38$;	$x=18580+7s$;	$s \sqsupset -2654\frac{4}{7}$,	2655.
Et c.	Et c.	Et c.	Et c.

And the 2072 term of this series will be

$$x=10368; x=4+7s; s \sqsupset -\frac{4}{7}; \text{ has } 1.$$

And, therefore, this whole series is composed of seven such arithmetical progressions, as the former.

				Values.
If $x=4$;	$x=18644-7s$;	$s=2663\frac{3}{4}$;	s has 2663 .	
$x=9$;	$x=18635-7s$;	$s=2662\frac{1}{2}$;		2662.
$x=14$;	$x=18626-7s$;	$s=2660\frac{1}{2}$;		2660.
$x=19$;	$x=18617-7s$;	$s=2659\frac{1}{2}$;		2659.
$x=24$;	$x=18608-7s$;	$s=2658\frac{1}{2}$;		2658.
$x=29$;	$x=18599-7s$;	$s=2657$;		2656.
$x=34$;	$x=18590-7s$;	$s=2655\frac{1}{2}$;		2655.
$x=39$;	$x=18581-7s$;	$s=2654\frac{1}{2}$;		2654.
Etc.	Etc.	Etc.	Etc.	

And the 2071 term of this series will be

$$x=10354; x=14-7s; s=2, s \text{ has 1 value.}$$

But if another term be taken, the value of s will be nothing. For if

$$x=10359; x=5-7s; s=0\frac{5}{7}, s \text{ has 0.}$$

Therefore the whole series may be considered as consisting of 7 such arithmetical progressions as the former.

Lastly,

				Values.
If $x=5$;	$x=18645-7s$;	$s=2663\frac{1}{4}$;	s has 2663 .	
$x=10$;	$x=18636-7s$;	$s=2662\frac{1}{2}$;		2662.
$x=15$;	$x=18627-7s$;	$s=2661$;		2660.
$x=20$;	$x=18618-7s$;	$s=2659\frac{1}{2}$;		2659.
$x=25$;	$x=18609-7s$;	$s=2658\frac{1}{2}$;		2658.
$x=30$;	$x=18600-7s$;	$s=2657\frac{1}{2}$;		2657.
$x=35$;	$x=18591-7s$;	$s=2655\frac{1}{2}$;		2655.
$x=40$;	$x=18582-7s$;	$s=2654\frac{1}{2}$;		2654.
Etc.	Etc.	Etc.	Etc.	

And the 2071 term of this series will be

$$x=10355; x=15-7s; s=2\frac{1}{7}, s \text{ has 2 values.}$$

And in the next term s will have no value. And therefore this series consists also of seven arithmetical progressions, such as the former.

Now

Now the sum of the 7 first terms of the progressions arising when	$x=1, 6, 11, \&c.$ is 18624;
ditto	$x=2, 7, 12, \&c.$ is 18618;
ditto	$x=3, 8, 13, \&c.$ is 18619;
ditto	$x=4, 9, 14, \&c.$ is 18613;
ditto	$x=5, 10, 15, \&c.$ is 18614;

Therefore the sum of the first terms of the 35 } is 93088.
progressions - - - - - }

And the whole number of answers, which can be given to the question, are comprized in 296 terms of an arithmetical progression, whose greatest term is 93088, and common difference $(3\frac{1}{2} \times 9 =) 31\frac{1}{2}$: The sum of which progression will, upon computation, appear to be 13801148, the same as the result of the former solution.

C O R O L L .

As in those equations, which have but two indetermined quantities, it seems sufficient to point out a few answers, and to determine their number; so in those, where there are three indetermined quantities, the question may be considered as solved, as soon as the arithmetical progressions that contain the number of answers are ascertained: Now, although from the different methods, by which the solution of those questions has been attempted, it appears, that it will be difficult to determine, from the given equation, without some kind of process, how many such progressions will be necessary; yet, from the last method, we may safely conclude, that their number can never exceed the product of the coefficients of x and y .

COROL. II.

' Equations, which contain four indetermined quantities, may be solved by the joint application of the methods above delivered: That is to say, by substituting 1, 2, 3, &c. for that indetermined quantity which has the greatest coefficient; and then applying either of the former methods to solve the resulting equations, in which there will be three indetermined quantities, as will be hereafter exemplified.

SCHOLIUM. Having before pointed out the cases, whose solutions may be most readily come at, in such questions which have but two indetermined numbers, they will assist in finding those which are easiest, when there are three numbers not determined.

Mr. *De Moivre*, in question 6, has assumed the coefficient of z to be the sum of the coefficients of x and y , and therein has given the easiest case possible, and the following questions are introduced to shew the effects of assuming for the coefficient of z a multiple of the coefficient of x , or of y .

QUESTION IX.

How many different answers in affirmative integers, will the equation $3x + 5y + 9z = 1849$, admit of?

SOLUTION.

$$\text{Here } x = \left(\frac{1849 - 5y - 9z}{3} \right) 616 - y - 3z + \frac{1 - 2y}{3}$$

Let

Let $r = \frac{1-2y}{3}$; Then $3r = 1-2y$;

Th. $y = \left(\frac{1-3r}{2} \right) = r + \frac{1-r}{2}$.

Let $s = \frac{1-r}{2}$; Then $2s = 1-r$;

Th. $r = 1-2s$;

$$y = \left(\frac{1-3 \times \frac{1-2s}{2}}{2} \right) = 3s-1;$$

And $x = \left(\frac{1849-5 \times 3s-1-9s}{3} \right) = 618-5s-3s$;

Where $s = \frac{1}{y}$, And $s = \left(\frac{618-3}{5} \right) = 103$.

Now in this case the values of x are all limited by the column which contains the value of x .

Values.

If $s=1$; $x=613-3s$; $y=3-1=2$; x has 204
 $s=2$; $x=608-3s$; $y=6-1=5$; x has 202
 $s=3$; $x=603-3s$; $y=9-1=8$; x has 200
 $s=4$; $x=598-3s$; $y=12-1=11$; x has 199
 &c. &c. &c. &c.

Therefore the values of x make three arithmetical progressions, 204, 199, 194, &c. 202, 197, 192, &c. and 200, 195, 190, &c. And because $s = 103$, each progression will consist of $\left(\frac{102}{3} \right) = 34$ terms.

QUESTION X.

How many different answers, in affirmative integers, will the equation $3x + 5y + 20z = 1849$, admit of?

SOLUTION.

$$\text{Here } x = \left(\frac{1849 - 5y - 20z}{3} = \right) 616 - y - 6z + \frac{1 - 2y - 2z}{3};$$

$$\text{Let } r = \frac{1 - 2y - 2z}{3}; \text{ Then } 3r = 1 - 2y - 2z;$$

$$\text{Th. } y = \left(\frac{1 - 2z - 3r}{2} = \right) z - r + \frac{1 - r}{2};$$

$$\text{Let } s = \frac{1 - r}{2}; \text{ Then } 2s = 1 - r;$$

$$\text{Th. } r = 1 - 2s;$$

$$y = \left(\frac{1 - 2z - 3 \times 1 - 2s}{2} = \right) 3s - z - 1;$$

$$\text{And } x = \frac{1849 - 15s + 5z + 5 - 20z}{3};$$

$$\text{Or } x = \left(\frac{1854 - 15s - 15z}{3} = \right) 618 - 5s - 5z;$$

Where (because the coefficients of s and z are equal)

Put $p = s + z$, or $s = p - z$:

Then $x = 618 - 5p$;

And $y = (3 \times p - z - z - 1) = 3p - 4z - 1$;

Where $p = \left(\frac{618}{5} = \right) 123 \frac{3}{5}$, and $p = \left(\frac{1}{5} = \right) 1 \frac{1}{5}$;

So that p will have 122 values.

And the values of z are all limited by the column which contains the values of y .

If $y=2$; $x=618-10=608$; $y=5-4x$; x has 1 value.
 $y=3$; $x=618-15=603$; $y=8-4x$; x 1 ditto.
 $y=4$; $x=618-20=598$; $y=11-4x$; x 2 values.
 $y=5$; $x=618-25=593$; $y=14-4x$; x 3 values.
 $y=6$; $x=618-30=588$; $y=17-4x$; x 4 ditto.
 $y=7$; $x=618-35=583$; $y=20-4x$; x 4 ditto.
 $\text{&c.} \qquad \qquad \text{&c.} \qquad \qquad \text{&c.} \qquad \qquad \text{&c.}$

Here the number of the values of x compose 4 arithmetical progressions, two of which are equal; viz. 1, 4, 7, 10, &c. 1, 4, 7, 10, &c. 2, 5, 8, 11, &c. and 3, 6, 9, 12, &c. And because $\frac{122}{4} = 30 \frac{2}{4}$, the two first will consist of 31 terms; and the two last of thirty.

Hence, *First*, In any question of this kind, if either of the lesser coefficients will measure the greater; the number of arithmetical progressions, necessary to its solution, will not exceed the number by which the said lesser coefficient measures the greater.

Secondly, If in any column, A or B, expressing the values of x or y , the coefficient of x be greater than the common difference of the progression, whereby the absolute numbers increase or decrease; then two or more of the arithmetical progressions, exhibiting the number of the values of x will be equal.

In the following question the coefficients of y and x are so assumed, that if they be severally divided by the coefficient of x , the same remainder may be left.

QUES.

QUESTION XI.

How many different answers, in affirmative integers, will the equation $3x + 5y + 17z = 1849$ admit of?

SOLUTION.

$$\text{Here } x = \left(\frac{1849 - 5y - 17z}{3} \right) = 616 - y - 5z + \frac{1 - 2y - 2z}{3} :$$

$$\text{Let } r = \frac{1 - 2y - 2z}{3} ; \text{ Then } 3r = 1 - 2y - 2z ;$$

$$\text{Th. } y = \left(\frac{1 - 2z - 3r}{2} \right) = -z - r + \frac{1 - r}{2} :$$

$$\text{Let } s = \frac{1 - r}{2} ; \text{ Then } 2s = 1 - r ;$$

$$\text{Th. } r = 1 - 2s ;$$

$$y = \left(\frac{1 - 2z - 3 \times 1 - 2s}{2} \right) = 3s - z - 1 ;$$

$$\text{And } x = \frac{1849 - 15s + 5z + 5 - 17z}{3}$$

$$\text{Or } x = \left(\frac{1854 - 15s - 12z}{3} \right) = 618 - 5s - 4z :$$

$$\text{Wh. } s = \frac{z}{5} ; \text{ And } s = \left(\frac{614}{5} \right) = 122 \frac{4}{5} ;$$

A

B

If $s=1$; $x=613-4z$; And $y=2-z$; z has 1 value.

$s=2$; $x=608-4z$; $y=5-z$; z has 4 values.

$s=3$; $x=603-4z$; $y=8-z$; z has 7 values.

&c.

&c.

&c.

&c.

If

If $s=122$; $x=8-4x$; And $y=365-x$; x has 1 value.			
$s=121$; $x=13-4x$;	$y=362-x$;	x	3 values.
$s=120$; $x=18-4x$;	$y=359-x$;	x	4
$s=119$; $x=23-4x$;	$y=356-x$;	x	5
$s=118$; $x=28-4x$;	$y=353-x$;	x	6
$s=117$; $x=33-4x$;	$y=350-x$;	x	8
$\&c.$	$\&c.$	$\&c.$	$\&c.$

Here the column B gives the arithmetical progression 1, 4, 7, 10, $\&c.$ only, for the numbers of the values of x ; but the column A produces the four progressions following, *viz.* 1, 6, 11, 16, $\&c.$ 3, 8, 13, 18, $\&c.$ 4, 9, 14, 19, $\&c.$ And 5, 10, 15, 20, $\&c.$

Hence, when the two greater coefficients, being severally divided by the lesser, leave the same remainder; then one of the columns will exhibit a single progression for the numbers of the values of x , while the same are limited thereby.

QUESTION XII.

It is required to find what number of different answers, in affirmative integers, may be given to the following equation, *viz.* $2x + 3y + 5z + 30w = 100003$.

SOLUTION.

Here (following the method used in the second solution of question 8).

Let $w = 1$; Then $2x + 3y + 5z + 30 = 100003$;

Th. $2x = 100003 - 30 - 3y - 5z$,

Or

$$\text{Or } x = \left(\frac{99973 - 3r - 5z}{2} \right) 49986 - y - 2x + \frac{1 - y - z}{2}$$

$$\text{Let } r = \frac{1 - y - z}{2}; \text{ Then } 2r = 1 - y - z;$$

$$\text{Th. } y = 1 - z - 2r;$$

$$\text{And } x = \frac{99973 - 3 \times 1 - z - 2r - 5z}{2},$$

$$\text{Or } x = \left(\frac{99973 - 3 + 3z + 6r - 5z}{2} \right) 49985 + 3r - z.$$

In the expression $y = 1 - z - 2r$, it is evident, that if r be nothing or affirmative, y will be negative; Let therefore the sign of r be changed in the values of both x and y ; And then,

$$y = r + 2r - z,$$

$$\text{And } x = 49985 - 3r - z;$$

$$\text{Th. } r = (1 - 1) = 0; \text{ And } r = \left(\frac{49984}{3} \right) 16661 \frac{1}{3}.$$

A

B

If $r = 1$; $x = 49982 - z$; $y = 3 - z$; z has 2 values.

$$r = 2; x = 49979 - z; y = 5 - z; z \quad 4$$

$$r = 3; x = 49976 - z; y = 7 - z; z \quad 6$$

Etc.

Etc.

Etc.

Etc.

$$r = 16661; x = 2 - z; y = 33323 - z; \text{ has 1 value.}$$

$$r = 16660; x = 5 - z; y = 33321 - z; \quad 4 \text{ values.}$$

$$r = 16659; x = 8 - z; y = 33319 - z; \quad 7$$

Etc.

Etc.

Etc.

Etc.

Now, in order to find the number of terms in which the value of z is limited by the column B,

$$\text{Let } 49982 - 3z = 3 + 2z;$$

$$\text{And } 5z = 49979; \text{ Th. } z = 9995 \frac{1}{5}.$$

There-

Therefore the column B limits the value of x in 9996 terms; and the column A in $(16661 - 9996 =) 6665$ terms.

Let $x = 2$; Then $2x + 3y + 5z + 60 = 100003$;

Th. $2x = 100003 - 60 - 3y - 5z$,

Or $x = \left(\frac{99943 - 3y - 5z}{2} \right) 49971 - 2y + \frac{1 - y - z}{2}$;

Which remainder being the same as before obtained,

Th. $\begin{cases} y = 1 + 2r - z; \text{ as in the former substitution,} \\ x = 49970 - 3r - z, \text{ differing from the former by } 15. \end{cases}$

$r \sqsubset 0$; And $r \sqsupset \left(\frac{49969}{3} \right) 16656 \frac{1}{3}$.

Now when $r = 1$, or 2, or 3, &c. x has 2, or 4, or 6 values, as before, and if

$r = 16656$; $x = 2 - z$; $y = 33313 - z$; has 1 value.

$r = 16655$; $x = 5 - z$; $y = 33311 - z$; has 4 values.

$r = 16654$; $x = 8 - z$; $y = 33309 - z$; has 7 ditto.

&c.

&c.

&c.

&c.

Also to find when the column B ceases to limit the value of z ,

$$49967 - 3x = 3 + z;$$

Th. $5x = 49964$. And $x = \frac{49964}{5} = 9992 \frac{4}{5}$

Therefore the column B limits the values of z in 9993 terms; and the column A in $(16656 - 9993 =) 6663$ terms.

If the above kinds of process be continued, by substituting 3, 4, 5, &c. for n , it will appear that when

$n=1$;	The number of the	9996	Terms; and the	6665	Terms.
$n=2$;	values of z will be two	9993	other will have	6663	
$n=3$;	arithmetical progres-	9990	one for its least	6661	
$n=4$;	sions, the first of which	9987	term, and 3 for	6659	
$n=5$;	will have 2 for its	9984	its common dif-	6657	
$n=6$;	least term and com-	9981	ference; and will	6655	
&c.	mon difference, and	&c.	consist of	&c.	
&c.	will consist of	&c.		&c.	

Or assuming n , according to its other limit,

$$n \square \left(\frac{100003 - 2 - 3 - 5}{30} = \frac{99993}{30} = \right) 3333 \frac{3}{30};$$

Then the two arithmetical progressions which contain the number of the values of z will, when

$n = 3333$	} be	{	0	Terms of the pro-	{	1	Terms of the pro-
$n = 3332$			3			3	
$n = 3331$			6			5	
$n = 3330$			9			7	
$n = 3329$			12			9	
$n = 3328$			15			11	
$n = 3327$			18			13	
&c.		&c.		&c.			&c.

Which may be concluded, without making the substitutions, by finding the 3333d, 3332d, &c. terms of the above given progressions.

Now the	Terms of the arithme-		
sum of	tical progression.		
0	$2+4+6;$	$= 0;$	
3	$2+4+6;$	$= 12;$	
6	$2+4+6+8+10+12;$	$= 12+30;$	
9	$2+4+6+8+10, \&c.$	$= 12+30+48;$	
12	$2+4+6+8+10, \&c.$	$= 12+30+48+66;$	
15	$2+4+6+8+10, \&c.$	$= 12+30+48+66+84$	
&c.	&c.	&c.	

And

And the sums are a rank of numbers, whose second differences are equal, as appears below.

Sums.	1st Differences.	2d Differences.
0.		
12.	12.	
12+30.	30.	18.
12+30+48.	48.	18.
12+30+48+66.	66.	18.
12+30+48+66+84.	84.	18.
Σc.	Σc.	Σc.

And the sum of 3333 terms of a series, whose first term is 0, the first of its first differences 12, and the first of its second differences 18, will (by Quest. 52. Part II. Vol. I.) be

$$3333 \times 0 + \frac{3333 \times 3332 \times 12}{1 \times 2} + \frac{3333 \times 3332 \times 3331 \times 18}{1 \times 2 \times 3}$$

Against the sum of	1	Terms of the	1;	=1;
3	1	1+3+5;	=1+8;	
5	1	1+3+5+7+9;	=1+8+16;	
7	1	1+3+5+7+9, &C.	=1+8+16+24;	
9	1	1+3+5+7+9, &C.	=1+8+16+24+32;	
&C.	&C.	&C.	&C.	

In which the sums are	1st Differences.	2d Differences.
1.		
1+8.	8.	
1+8+16.	16.	8.
1+8+16+24.	24.	8.
1+8+16+24+32.	32.	8.

And the sum of 3333 terms, thereof, will be

$$3333 \times 1 + \frac{3333 \times 3332 \times 8}{1 \times 2} + \frac{3333 \times 3332 \times 3331 \times 8}{1 \times 2 \times 3}$$

And,

And, because the number of the terms of both series is the same, the sum of them both will be, $3333 \times 0 + 1$

$$+ \frac{3333 \times 3332}{1 \times 2} \times 12 + 8 + \frac{3333 \times 3332 \times 3331}{1 \times 2 \times 3} \times 18 + 8;$$

That is 160190378249.

From the above process, it appears that it would have been more convenient to have began the substitution with the greatest value of n ; because then, the least terms of the arithmetic series, whose sums are the answer, would have been immediately produced.

QUESTION XIII.

Let there be a series formed in the following manner, viz. Let the first term of it be the sum of n numbers in arithmetical progression, the second term, the sum of $n + m$ numbers; the third term, the sum of $n + 2m$ numbers; the fourth term, the sum of $n + 3m$ numbers of the same progression, &c. then the sum of p terms of this series is required?

SOLUTION.

Let the given arithmetical progression be represented by $a, a + d, a + 2d, a + 3d$, &c. Then will

n terms of $a, a + d$, &c. be $na + \frac{n(n-1)}{2} d;$

$n + m$

$$\begin{array}{lll}
 n+m \text{ terms thereof } na + ma + \frac{nn+2nm+mm-n-m}{2}d; & & \\
 n+2m \text{ terms } na + 2ma + \frac{nn+4nm+4mm-n-2m}{2}d; & & \\
 n+3m \text{ terms } na + 3ma + \frac{nn+6nm+9mm-n-3m}{2}d; & & \\
 \text{\textit{Etc.}} & \text{\textit{Etc.}} & \text{\textit{Etc.}}
 \end{array}$$

The first differences of which
are

And their second
differences.

$$\begin{array}{lll}
 ma + \frac{2nm+mm-m}{2}d. & & \\
 ma + \frac{2nm+3mm-m}{2}d. & & mm d \\
 ma + \frac{2nm+5mm-m}{2}d. & & mm d \\
 \text{\textit{Etc.}} & \text{\textit{Etc.}} & \text{\textit{Etc.}}
 \end{array}$$

And therefore (by Quest. 52. Part II. Vol. I.) the
sum of p terms of the series will be,

$$\begin{aligned}
 p \times na + \frac{nn-n}{2}d + \frac{p \cdot p - 1}{2} \times ma + \frac{2nm+mm-m}{2}d. \\
 \left(+ \frac{p \cdot p - 1 \cdot p - 2}{1 \cdot 2 \cdot 3} mm d, \right. \\
 \text{Or (if } P = \frac{p-1 \times m}{2} \text{)} \text{ it will become} \\
 \hline
 n + P \times a + \frac{2n+m-1}{2} + \frac{p-2 \times m}{3} \times P + \frac{n \cdot n - 1}{1 \cdot 2} \times d \times p.
 \end{aligned}$$

EXAMPLE. If the arithmetical progression be 3, 5, 7,
9, 11, 13, &c. and therefrom be formed a series, 24,
48, 80, 120, 168, 224, 288, &c. by taking the sum
of

of the four first numbers for the first term, and for the remaining terms the sum of 6, 8, 10, 12, &c. numbers of the same progression. Then let the sum of 7 terms of this series be required.

Here $a = 3$; $d = 2$; $n = 4$; $m = 2$ and $p = 7$;

Then $\frac{6 \times 2}{2} = 6 = P$; $\frac{4 + 6}{2} \times 3 = 30$; $\frac{2 \times 4 + 2 - 1}{2} = \frac{5}{2}$;

$\frac{7 - 2 \times 2}{3} = \frac{10}{3}$; $\frac{4 \times 3}{2} = 6$;

$\frac{5}{2} + \frac{10}{3} \times 6 = (27 + 20 =) 47$; And $47 + 6 = 53$;

$53 \times 2 = 106$; And $106 + 30 = 136$; Lastly $136 \times 7 = 952$.
The sum required.

QUESTION XIV.

How many different values can x , y , z and u have, in affirmative integers, in the equation $3x + 5y + 19z + 143u = 91306$?

Method of SOLUTION.

Here $u = 638 \frac{45}{143}$.

If $u = 638$; Then $3x + 5y + 19z + 91324 = 91306$.

Or $3x + 5y + 19z = 72$;

Whence $x = \left(\frac{72 - 5y - 19z}{3} \right) 24 - y - 6z - \frac{2y + z}{3}$.

Let $r = \frac{2y + z}{3}$; Then $2y = 3r - z$;

Th.

Th. $y = \left(\frac{3r-z}{2} \right) r + \frac{r-z}{2}$.

Let $s = \frac{r-z}{2}$; And $2s = r-z$;

Th. $r = 2s + z$,

$$y = \left(\frac{3 \times 2s + z - z}{2} = \frac{6s + 3z - z}{2} = \right) 3s + z;$$

And $x = 24 - 5s - 8z$;

And putting $s = p - z$; Then $y = 3p - 2z$; Where $p = \frac{2}{3}$;

And $x = 24 - 5p - 3z$; Where $p = \left(\frac{24-3}{5} = \frac{21}{5} = \right) 4\frac{1}{5}$.

If $p=1$;	Then $x=19-3z$;	And $y=3-2z$;	z has 1	} Values.
$p=2$;	$x=14-3z$;	$y=6-2z$;	z 2	
$p=3$;	$x=9-3z$;	$y=9-2z$;	z 3	
$p=4$;	$x=4-3z$;	$y=12-2z$;	z 4	

Where the values of z are limited by each of the columns A and B for two terms.

If $u = 637$; Then $3x + 5y + 19z + 91091 = 91306$,
Or $3x + 5y + 19z = 215$;

Whence $x = \left(\frac{215 - 5y - 19z}{3} \right) 71 - y - 6z + \frac{2-2y-z}{3}$.

Let $r = \frac{2-2y-z}{3}$; Then $2y = z - z - 3r$,

Th. $y = \left(\frac{z-z-3r}{2} \right) 1 - \frac{z+r}{2}$;

Let $s = \frac{z+r}{2}$; Then $2s = z+r$; And $2s - z = r$;

Then $y = \left(\frac{z-z-3 \times 2s-z}{2} \right) 1 + z - 3s$,

And

$$\text{And } x = \left(\frac{215 - 5 \times \overline{1+z-3p-19z}}{3} \right) 76 + 5p - 8z.$$

Wherein (putting $s = z - p$) there arises

$$y = (1 + z - 3 \times z - p) = 3p + 1 - 2z,$$

$$\text{And } x = (70 + 5 \times z - p - 8z) = 70 - 5p - 3z;$$

$$\text{Where } p = \left(\frac{2 \times 1 - 1}{3} \right) = \frac{1}{3}; \text{ And } p = 13\frac{2}{3}.$$

If $p = 1$; $x = 65 - 3z$; And $y = 4 - 2z$; z has	1	} Values.
$p = 2$; $x = 60 - 3z$; $y = 7 - 2z$; z has	3	
$p = 3$; $x = 55 - 3z$; $y = 10 - 2z$; z has	4	
$p = 4$; $x = 50 - 3z$; $y = 13 - 2z$; z has	6	
$p = 5$; $x = 45 - 3z$; $y = 16 - 2z$; z has	7	
$p = 6$; $x = 40 - 3z$; $y = 19 - 2z$; z has	9	
$p = 7$; $x = 35 - 3z$; $y = 22 - 2z$; z has	10	
$p = 8$; $x = 30 - 3z$; $y = 25 - 2z$; z has	9	
$p = 9$; $x = 25 - 3z$; $y = 28 - 2z$; z has	8	
$p = 10$; $x = 20 - 3z$; $y = 31 - 2z$; z has	6	
$p = 11$; $x = 15 - 3z$; $y = 34 - 2z$; z has	4	
$p = 12$; $x = 10 - 3z$; $y = 37 - 2z$; z has	3	
$p = 13$; $x = 5 - 3z$; $y = 40 - 2z$; z has	1	

Where the values of z are limited by the column B for 7 terms; and by the column A for the remaining 6 terms.

$$\text{If } x = 636; \text{ Then } 3x + 5y + 19z + 90948 = 91306,$$

$$\text{Or } 3x + 5y + 19z = 358;$$

$$\text{Whence } x = \left(\frac{358 - 5y - 19z}{3} \right) 119 - y - 6z + \frac{1 - 2y - z}{3};$$

Let

$$\text{Let } r = \frac{1-2y-z}{3}; \text{ Then } 2y = 1-z-3r;$$

$$\text{Th. } y = \left(\frac{1-z-3r}{2} \right) - r + \frac{1-z-r}{2};$$

$$\text{Let } s = \frac{1-z-r}{2}; \text{ Then } 2s = 1-z-r,$$

$$\text{And } r = 1-z-2s;$$

$$\text{Now } y = \left(\frac{1-z-3 \times 1-z-2s}{2} \right) \frac{1-z-3+3z+6s}{2},$$

$$\text{Or } y = \left(\frac{6s+2z-2}{2} \right) 3s+z-1;$$

$$\text{Lastly } x = \frac{358-15s-5z+5-19z}{3},$$

$$\text{Or } x = \left(\frac{363-15s-24z}{3} \right) 121-5s-8z.$$

Where, because z has different signs in the values of x and y , let us substitute $s = p - z$;

$$\text{Then } x = (121-5 \times p-z-8z) 121-5p-3z;$$

$$\text{And } y = (3 \times p-z+z-1) 3p-2z-1;$$

$$\text{Where } p = \left(\frac{2 \times 1+1}{3} = \frac{3}{3} \right) 1; \text{ And } p = 23 \frac{2}{3};$$

And there are 22 values of p in the whole.

$\begin{array}{l} \text{If } p=2; \text{ Then } x=111-3z; \text{ And } y=5-2z; z \text{ has } 2 \\ p=3; \quad x=106-3z; \quad y=8-2z; z \text{ has } 3 \\ p=4; \quad x=101-3z; \quad y=11-2z; z \text{ has } 5 \\ \text{Etc.} \quad \quad \quad \text{Etc.} \quad \quad \quad \text{Etc.} \quad \text{Etc.} \end{array}$	}	Values.
--	---	---------

If $p=23$; Then $x=6-3z$; And $y=68-2z$; z has 1

$p=22$;	$x=11-3z$;	$y=65-2z$;	z has 3	} Values
$p=21$;	$x=16-3z$;	$y=62-2z$;	z has 5	
$p=20$;	$x=21-3z$;	$y=59-2z$;	z has 6	
£c.	£c.	£c.	£c.	

Also $\frac{111-5n}{3} = \frac{5+3n}{2}$; Or $222-10n = 15+9n$,

Or $222-15 = 10n+9n$; Th $n = \left(\frac{207}{19}\right) 10 \frac{17}{19}$;

Therefore the values of z are limited by the column B for the first 11 terms, and by the column A in the last.

If $u = 635$, Then $3x+5y+19z+90805=91306$,

Or $3x+5y+19z=501$;

Whence $x = \left(\frac{501-5y-19z}{3}\right) = 167-y-6z-\frac{2y+z}{3}$.

And, by proceeding in the same manner as when u was assumed $= 638$, it will appear

That $y = 3s + z$;

And $z = \left(\frac{501-5 \times 3s + z - 19z}{3}\right) = 167-5s-8z$;

Or, putting $s = p - z$,

$y = 3p - 2z$; Where $p \sqsubset \frac{2}{3}$;

And $x = 167-5p-3z$; And $p \sqsupset \left(\frac{167-3}{5}\right) 32\frac{4}{5}$.

Here it is evident, that the value of y (and the lesser limit of p which depends thereon) are the same with those found when $u = 638$; Also that the value of x , here found, is greater than the value of x , when $u = 638$, by 143, the coefficient of u ; And consequently the
greater

greater limit of p exceeds the former, by $\left(\frac{143}{5} =\right)$
 $28\frac{3}{5}$.

If $p=1$; Then $x=162-3z$; And $y=3-2z$; z has 1			} Values.
$p=2$;	$x=157-3z$;	$y=6-2z$; z has 2	
$p=3$;	$x=152-3z$;	$y=9-2z$; z has 4	
$\&c.$	$\&c.$	$\&c.$	

If $p=32$; Then $x=7-3z$; And $y=96-2z$; z has 2			} Values.
$p=31$;	$x=12-3z$;	$y=93-2z$; z has 3	
$p=30$;	$x=17-3z$;	$y=90-2z$; z has 5	
$p=29$;	$x=22-3z$;	$y=87-2z$; z has 7	
$\&c.$	$\&c.$	$\&c.$	$\&c.$

Hence, so long as the values of z are limited by the column B, the arithmetical progressions, expressing those values, are the same with those which arose when $z=638$, the number of terms excepted.

$$\text{Now } \frac{162-5z}{3} = \frac{3+3z}{2}; \text{ Or } 324-10z=9+9z;$$

$$\text{Or } 324-9=10z+9z; \text{ Th. } z=\left(\frac{315}{19}\right) 16\frac{11}{19}.$$

In this case, therefore, the values of z are limited by the column B for 17 terms; and for the remaining $(32-17=)$ 15 terms, by the column A.

From this last process, (wherein 635 the value of z differs from 638, the first assumed value thereof, by 3 the coefficient of z) it may be safely concluded, that when z is 632, 629, 626, $\&c.$ the expression of the value of y will always be $3p-2z$, as in the two assumptions above quoted; And the expressions of the value of x will constantly differ by 143, the coefficient of z .

D 2

Hence,

Hence, when $n = 632$; Then

$$\begin{array}{lcl}
 \text{If } p=1; & x=305-3x; & \text{And } y=3-2x; \quad x \text{ has } 1 \\
 p=2; & x=300-3x; & y=6-2x; \quad x \text{ has } 2 \\
 p=3; & x=295-3x; & y=9-2x; \quad x \text{ has } 4
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{Values.}$$

$\mathcal{E}c.$ $\mathcal{E}c.$ $\mathcal{E}c.$ $\mathcal{E}c.$

When $n = 629$,

$$\begin{array}{lcl}
 \text{If } p=1; & x=448-3x; & \text{And } y=3-2x; \quad x \text{ has } 1 \\
 p=2; & x=443-3x; & y=6-2x; \quad x \text{ has } 2 \\
 p=3; & x=438-3x; & y=9-2x; \quad x \text{ has } 4
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{Values.}$$

$\mathcal{E}c.$ $\mathcal{E}c.$ $\mathcal{E}c.$ $\mathcal{E}c.$

And, therefore, in all the assumptions of n , that are contained in the series 638, 635, 632, 629, $\mathcal{E}c.$ the same arithmetical progressions shew how many values of x are possible, when those values are limited by the column B; viz. The progressions 1, 4, 7, 10, $\mathcal{E}c.$ and 2, 5, 8, 11, $\mathcal{E}c.$ But the number of the terms in each progression are different in every particular substitution: Thus, when $n = 638$, there was but one term of each progression useful; when $n = 638$, then, because the column B limits the values of x for 17 terms; therefore 9 terms of the first progression, and 8 terms of the second are useful.

It remains to find the number of terms that will be useful, in the succeeding substitutions:

When

$$\text{When } n=632, \text{ Then } \frac{305-5n}{3} = \frac{3+3n}{2};$$

$$\text{Th. } 610-10n = 9+9n,$$

$$\text{Or } 610-9 = 10n+9n;$$

$$\text{Th. } \frac{601}{19} = n;$$

$$\text{That is } 31\frac{12}{19} = n.$$

Therefore the column B will limit the values of n for 31 terms, after the first, that is for 32 terms; and, consequently, there will be 16 terms of each progression useful.

By comparing the last process, with that used when 635 was substituted for n ; it will appear, that the values of n differ by $\left(\frac{143 \times 2}{19} = \frac{286}{19} = 15\frac{1}{19}\right)$; which will also be the case in all succeeding substitutions for n , that differ by 3.

D 3

There:

Therefore when

$u=638; n=1\frac{10}{19};$	And the number of the terms, for which the column	2;	1	1
$u=635; n=16\frac{11}{19};$	B limits the values of z , will be	17;	9	8
$u=632; n=31\frac{12}{19};$		32;	16	16
$u=629; n=46\frac{13}{19};$		47;	24	23
$u=626; n=61\frac{14}{19};$		62;	31	31
$u=623; n=76\frac{15}{19};$		77;	39	38
$u=620; n=91\frac{16}{19};$		92;	46	46
$u=617; n=106\frac{17}{19};$		107;	54	53
$u=614; n=121\frac{18}{19};$		122;	61	61
$u=611; n=137$		138;	69	69
$u=608; n=152\frac{1}{19};$		153;	77	76
$u=605; n=167\frac{2}{19};$		168;	84	84
$u=602; n=182\frac{3}{19};$		183;	92	91
$u=599; n=197\frac{4}{19};$		198;	99	99
$u=596; n=212\frac{5}{19};$		213;	107	106
$u=593; n=227\frac{6}{19};$		228;	114	114
$u=590; n=242\frac{7}{19};$		243;	122	121
$u=587; n=257\frac{8}{19};$		258;	129	129
$u=584; n=272\frac{9}{19};$		273;	137	136
$u=581; n=287\frac{10}{19};$		288;	144	144
£c.	£c.	£c.	£c.	£c.

Therefore

And

Terms of the progression, 1, 4, 7, 10, £c.

Terms of the progression, 2, 5, 8, 11, £c. will be useful.

Hence, when the numbers, substituted for u , differ by $(3 \times 19 =) 57$, then the number of useful terms, in each of the arithmetical progressions, which express the values of z , will differ by 143, the coefficient of u .

That is to say, when the terms of the series, 638, 581, 524, 467, £c. are severally substituted for u , Then both the arithmetical progressions, expressing the values of z , will contain 1, 144, 287, 430, £c. terms; when the terms of the series 635, 578, 521, 464, £c. are severally substituted for u , then the arithmetical progression, 1, 4, 7, 10, £c. must be continued to 9, 152, 295, 438, £c. terms; and the other progression 2, 5, 8, 11, £c. to 8, 151, 294, 437, £c. terms; and so on for the rest.

Now,

Now, because $\left(\frac{629}{57}\right) 11 \frac{2}{57}$, the four first of those pairs of series consist of 11 terms, and the other fifteen of 10 terms each, all which may be separately summed by Quest. 13.

Or let the respective terms of the two arithmetical progressions be added together, and they will compose the progression 3, 9, 15, 21, &c. Of which the number of useful terms, in each of the substitutions for x , viz. 638, 635, 632, 629, &c. will be half the number of terms, for which the column B limits the values of x , viz. 1, $\frac{17}{2}$, 16, $\frac{47}{2}$, &c. Or, because these numbers depend upon the values of x , which have been proved to differ by $\left(15 \frac{1}{19} \text{ or } \right) \frac{286}{19}$, we may take the half of that, viz. $\frac{143}{19}$, for the difference of the number of useful terms in each substitution.

Whence, all the answers to the question, which can happen when the terms of the series 638, 635, 632, &c. are severally substituted for x , and the number of the values of x are limited by the column B, will nearly consist of a series formed from the arithmetical progression 3, 9, 15, 21, &c. by taking the first term of that progression, for the first term of the series; the sum of $1 + \frac{143}{19}$ terms of that progression, for the second term of the series; the sum of $1 + 2 \times \frac{143}{19}$ terms of that progression, for the third term of the series, &c. which series is summable by Quest. 13.

If $u=634$; Then $3x + 5y + 19z + 90662 = 91306$,

$$\text{Or } 3x + 5y + 19z = 644;$$

$$\text{Whence } x = \left(\frac{644 - 5y - 19z}{3} \right) 214 - y - 6z + \frac{2 - 2y - z}{3};$$

Therefore, (by proceeding as when 637 was substituted for u) $y = 1 + z - 3s$; And $x = 213 + 5s - 8z$;

$$\text{Or (putting } s = z - p)$$

$$y = 3p + 1 - 2z; \text{ And } x = 213 - 5p - 3z;$$

$$\text{Where } p \sqsubset \frac{1}{3} \quad \text{And } p \sqsupset \left(\frac{210}{5} \right) 42.$$

And here the value of y is the same, with that found when 637 was substituted for u , and the value of x exceeds its value, there found, by 143, in the same manner, as was observed before, concerning the results, when the numbers 638 and 635 were, severally, substituted for u .

$$\left. \begin{array}{llll} \text{If } p=1; x=208-3z; y=4-2z; & x \text{ has } 1 \\ p=2; x=203-3z; y=7-2z; & x \text{ has } 3 \\ p=3; x=198-3z; y=10-2z; & x \text{ has } 4 \end{array} \right\} \text{Values.}$$

$$\begin{array}{cccc} \text{Etc.} & \text{Etc.} & \text{Etc.} & \text{Etc.} \end{array}$$

$$\left. \begin{array}{llll} p=41; x=8-3z; y=124-2z; & x \text{ has } 2 \\ p=40; x=13-3z; y=121-2z; & x \text{ has } 4 \\ p=39; x=18-3z; y=118-2z; & x \text{ has } 5 \\ p=38; x=23-3z; y=115-2z; & x \text{ has } 7 \end{array} \right\} \text{Values.}$$

$$\begin{array}{cccc} \text{Etc.} & \text{Etc.} & \text{Etc.} & \text{Etc.} \end{array}$$

And therefore, so long as the values of z are limited by the column B, the arithmetical progressions, expressing those values, are the same with those which arose when $u=637$.

Now

$$\text{Now } \frac{208-5n}{3} = \frac{4+3n}{2}; \text{ Or } 416-10n=12+9n;$$

$$\text{Or } 416-12=10n+9n; \text{ Th. } n = \frac{404}{19} = 21 \frac{5}{19}.$$

In this case, therefore, the values of n are limited by the column B for 22 terms, and by the column A for the remaining $(41-22=)$ 19 terms.

Hence, therefore, we may conclude, as before, that in all the substitutions, for n , that are contained in the series 637, 634, 631, 628, &c. the same arithmetical progression will shew how many values of n are possible, while the number of those values is limited by the column B; and that the number of terms useful in each

progression will differ by $\frac{286}{19}$. Therefore, when:

$n=637; n=615;$	7;	4	3
$n=634; n=215;$	22;	11	11
$n=631; n=365;$	37;	19	18
$n=628; n=515;$	52;	26	26
$n=625; n=665;$	67;	34	33
$n=622; n=815;$	82;	41	41
$n=619; n=965;$	97;	49	48
$n=616; n=1115;$	112;	56	56
$n=613; n=1265;$	127;	64	63
$n=610; n=1415;$	142;	71	71
$n=607; n=1565;$	157;	79	78
$n=604; n=1715;$	172;	86	86
$n=601; n=1865;$	187;	94	93
$n=598; n=2015;$	202;	101	101
$n=595; n=2165;$	217;	109	108
$n=592; n=232$	233;	117	116
$n=589; n=247\frac{1}{19};$	248;	124	124
$n=586; n=262\frac{2}{19};$	263;	132	131
$n=583; n=277\frac{3}{19};$	278;	139	139
$n=580; n=292\frac{4}{19};$	293;	147	146
&c.	&c.	&c.	&c.

And the number of the terms for which the column B limits the values of n , will be

Therefore

Terms of the progression, 1, 4, 7, 10, &c.

And

Terms of the progression, 3, 6, 9, 12, &c. will be useful.

Whence the number of answers, of which the question is capable, when the terms of the series 637, 634, &c. are severally substituted for u , and the number of the values of z are limited by the column B, may be obtained, either exactly, or by approximation, in the manner above shewn.

If $u = 633$; Then $3x + 5y + 19z + 90519 = 91306$,

$$\text{Or } 3x + 5y + 19z = 787.$$

Where $x = \left(\frac{787 - 5y - 19z}{3} \right) 26z - y - 6x + \frac{1 - 2y - z}{3}$;

Whence (proceeding as when 636 was substituted for u)

$$x = 264 - 5z - 8z; \text{ And } y = 3z + z - 1; \text{ Or (if } z = p - z)$$

$$x = 264 - 5p - 3z; \text{ And } y = 3p - 2z - 1;$$

Where $p \sqsubset 1$; And $p \sqsupset \left(\frac{261}{5} \right) 52 \frac{1}{5}$.

And p will have 51 values; Therefore,

$$\left. \begin{array}{l} \text{If } p=2; x=254-3z; y=5-2z; z \text{ has } 2 \\ p=3; x=249-3z; y=8-2z; z \text{ has } 3 \\ p=4; x=244-3z; y=11-2z; z \text{ has } 5 \end{array} \right\} \text{Values.}$$

$$\text{\textit{Etc.} \quad \text{\textit{Etc.} \quad \text{\textit{Etc.} \quad \text{\textit{Etc.}}$$

$$\left. \begin{array}{l} p=52; x=4-3z; y=155-2z; z \text{ has } 1 \\ p=51; x=9-3z; y=152-2z; z \text{ has } 2 \\ p=50; x=14-3z; y=149-2z; z \text{ has } 4 \\ p=49; x=19-3z; y=146-2z; z \text{ has } 6 \end{array} \right\} \text{Values.}$$

$$\text{\textit{Etc.} \quad \text{\textit{Etc.} \quad \text{\textit{Etc.} \quad \text{\textit{Etc.}}$$

And here, it does not only happen, as before, that, so long as the values of z are limited by the column B, the arithmetical progressions, expressing those values, are the same with those found when $u=636$; but also, that when they are limited by the column A, then the progressions,

gressions, expressing those values, will be the same with those found when $u=638$, which number (638) exceeds the present substitution (633) by 5, which is the coefficient of y , and the common difference of the numbers in the column A. Hence,

First, we may determine the remainder of the values of z , when limited by the column B; in the same manner as before, *viz.*

If $u=636$; $n=101\frac{17}{19}$	11;	6	5
$u=633$; $n=25\frac{18}{19}$	26;	13	13
$u=630$; $n=41$	42;	21	21
$u=627$; $n=56\frac{19}{19}$	57;	29	28
$u=624$; $n=71\frac{19}{19}$	72;	36	36
$u=621$; $n=86\frac{19}{19}$	87;	44	43
$u=618$; $n=101\frac{4}{19}$	102;	51	51
$u=615$; $n=116\frac{5}{19}$	117;	59	58
$u=612$; $n=131\frac{6}{19}$	132;	66	66
$u=609$; $n=146\frac{7}{19}$	147;	74	73
$u=606$; $n=161\frac{8}{19}$	162;	81	81
$u=603$; $n=176\frac{9}{19}$	177;	89	88
$u=600$; $n=191\frac{10}{19}$	192;	96	96
$u=597$; $n=206\frac{11}{19}$	207;	104	103
$u=594$; $n=221\frac{12}{19}$	222;	111	111
$u=591$; $n=236\frac{13}{19}$	237;	119	118
$u=588$; $n=251\frac{14}{19}$	252;	126	126
$u=585$; $n=266\frac{15}{19}$	267;	134	133
$u=582$; $n=281\frac{16}{19}$	282;	141	141
$u=579$; $n=296\frac{17}{19}$	297;	149	148
<i>&c.</i>	<i>&c.</i>	<i>&c.</i>	<i>&c.</i>

And the number of the terms for which the column

B limits the values of z , will be

Therefore

Terms of the progression, 2, 5, 8, 11, &c.

And

Terms of the progression, 3, 6, 9, 12, &c. will be useful.

And hence, the number of answers of which the question is capable, when the terms of the series, 636, 633, &c. are severally substituted for u , and the number of the values of z are limited by the column B, may be determined as before.

In order to determine the number of answers, when the values of x are limited by the column A, we must first find how many values p is capable of. When

$x=638$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 4\frac{1}{3}$	And p has 4
$x=637$	$p \sqsubset \frac{1}{3}$	$p \sqsubset 13\frac{2}{3}$	And p has 13
$x=636$	$p \sqsubset 1$	$p \sqsubset 23\frac{1}{3}$	And p has 22
$x=635$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 32\frac{2}{3}$	And p has 32
$x=634$	$p \sqsubset \frac{1}{3}$	$p \sqsubset 42$	And p has 41
$x=633$	$p \sqsubset 1$	$p \sqsubset 52\frac{1}{3}$	And p has 51
$x=632$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 61\frac{2}{3}$	And p has 61
$x=631$	$p \sqsubset \frac{1}{3}$	$p \sqsubset 70\frac{1}{3}$	And p has 70
$x=630$	$p \sqsubset 1$	$p \sqsubset 80\frac{2}{3}$	And p has 79
$x=629$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 90$	And p has 89
$x=628$	$p \sqsubset \frac{1}{3}$	$p \sqsubset 99\frac{1}{3}$	And p has 99
$x=627$	$p \sqsubset 1$	$p \sqsubset 109\frac{2}{3}$	And p has 108
$x=626$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 118\frac{1}{3}$	And p has 118
$x=625$	$p \sqsubset \frac{1}{3}$	$p \sqsubset 127\frac{2}{3}$	And p has 127
$x=624$	$p \sqsubset 1$	$p \sqsubset 138$	And p has 136
$x=623$	$p \sqsubset \frac{2}{3}$	$p \sqsubset 147\frac{1}{3}$	And p has 147
$\&c.$	$\&c.$	$\&c.$	$\&c.$

Values,

Whence we may conclude, that when for x , are substituted the terms of the series 638, 623, 608; 593, $\&c.$ decreasing by $(3 \times 5 =) 15$; then the numbers of the values of p will be the corresponding terms of the series, 4, 147, 290, 433, $\&c.$ encreasing by 143.

And, when x is severally equal to 637, 622, 607, 592, $\&c.$ then the numbers of the values of p will be, respectively, 13, 156, 299, 442, $\&c.$ And so on.

Now since the arithmetical progressions, expressing the values of x (when those values are limited by the column A, and the terms of the series 638, 633, 628, 623, $\&c.$

$\&c.$ are severally substituted for u) are 1, 6, 11, 16, $\&c.$ 2, 7, 12, 17, $\&c.$ And 4, 9, 14, 19, $\&c.$ Therefore,

When			
$u=638;$	4-	$2=$	$2;$
$u=633;$	51-	$26=$	$25;$
$u=628;$	99-	$52=$	$47;$
$u=623;$	147-	$77=$	$70;$
$u=618;$	194-	$102=$	$92;$
$u=613;$	242-	$127=$	$115;$
$u=608;$	290-	$153=$	$137;$
$u=603;$	337-	$177=$	$160;$
$u=598;$	385-	$202=$	$183;$
$u=593;$	433-	$228=$	$205;$
$u=588;$	480-	$252=$	$228;$
$u=583;$	528-	$278=$	$250;$
$u=578;$	576-	$303=$	$273;$
$u=573;$	623-	$328=$	$295;$
$u=568;$	671-	$353=$	$318;$
$u=563;$	719-	$378=$	$341;$
$u=558;$	766-	$403=$	$363;$
$u=553;$	814-	$428=$	$386;$
$u=548;$	862-	$454=$	$408;$
$u=543;$	909-	$478=$	$431;$
$\&c.$			
Then $p-u$, that is to say the number of the terms in which the column A limits the values of z , will be.			
Therefore			
	1	2	3
Terms of the progression, 1, 6, 11, 16, $\&c.$	1	9	16
	1	8	16
Terms of the progression, 2, 7, 12, 17, $\&c.$ And	1	23	30
	1	16	23
Terms of the progression, 4, 9, 14, 19, $\&c.$ will be useful.	1	15	23
	1	8	15

Hence, when the terms of the series 638, 543, 448, 353, $\&c.$ decreasing by $(5 \times 19 =) 95$, are severally substituted for u , then the two first of the arithmetical progressions, expressing the numbers of the values of z , will severally contain, 1, 144, 287, 430, $\&c.$ terms, and the other arithmetical progression will contain, 0, 143, 286, 429, $\&c.$ terms.

And

And when, for x , the terms of the series, 633, 538, 443, 348, &c. are substituted; then the first of those arithmetical progressions will contain, 9, 152, 295, 438, &c. terms; and the remaining two will, each, contain 8, 151, 294, 437, &c. terms; and so on.

From what has been said, the number of answers which the question is capable of, while, for x , are severally substituted the terms of the series, 638, 633, 628, &c. and the values of x are limited by the column A, may be determined either exactly or nearly, by quest. 13.

We might now proceed to find the series that would result from substituting, 637, 632, 627, 622, &c. severally for x ; but, as the method above shewn, is very easy, and a sufficient number of examples have been given thereto, it is left for the reader's practice.

C O R O L.

From the whole it may be concluded, that the number of answers, in affirmative integers, which an equation, containing never so many indetermined quantities, is capable of, may be found, if the labour necessary thereto be not a hindrance thereof, by pursuing methods similar to the above: And 'tis hoped, that the reader will excuse the breaking off this question here, without either pointing out the four remaining classes of series, which will result while the numbers of the values of x are limited by the column A, or giving the sums of any one series in figures, as the reasons of so doing, are the fear of growing tedious, and the want of an artifice to shorten the operations.

SCHOLIUM.

SCHOLIUM. Question 12, the first of those which contain four indeterminated quantities, is of the easiest kind possible; the coefficient of z , being the sum of the coefficients of x and y ; and the coefficient of u , being a multiple of all the other coefficients: And the last question is of the most difficult kind that can happen, requiring the summation of $(3+5 \times 19=)$ 152 series of the second Order in its solution: It would be easy enough to enumerate all the properties, which the coefficients of the indeterminated quantities must have, to render the solution more difficult than the first, and less difficult than the last; but as the greatest part of them may be collected from what was said before, concerning equations which have but three indeterminated quantities, it will be needless to add more than the following, *viz.* that when the coefficient of u is a multiple of the coefficient of z , then the number of those series, whose second differences are equal, will be, as few as the number of arithmetical progressions would have been, if the equation had but three indeterminated quantities.

QUESTION XV.

The value of n terms of the series $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c.$ is required.

SOLUTION.

If $\frac{z}{r-1} = \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}, \&c. \text{ ad infinitum},$

Then it will appear that $z = 1.$

For,

For, if $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4}, \&c.$ be multiplied
by $r - 1$

$$\begin{array}{r} 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. \\ - \frac{1}{r} - \frac{1}{r^2} - \frac{1}{r^3}, \&c. \\ \hline \text{Product} = 1 + 0 + 0 + 0, \&c. \end{array}$$

Therefore $\frac{1}{r-1}$ is the value of the series, if it be infinitely continued.

But since the sum of n terms, only, are required, it will be necessary to subtract the value of all the terms after the n th term, from the value above found.

Now the terms, which follow the n th term, are $\frac{1}{r^{n+1}} + \frac{1}{r^{n+2}} + \frac{1}{r^{n+3}}, \&c. \text{ ad infinitum}$, which
(because $\frac{1}{r^{n+1}} = \frac{1}{r^n} \times \frac{1}{r}$; $\frac{1}{r^{n+2}} = \frac{1}{r^n} \times \frac{1}{r^2}, \&c.$)

will become $\frac{1}{r^n} \times \frac{1}{r} + \frac{1}{r^n} \times \frac{1}{r^2} + \frac{1}{r^n} \times \frac{1}{r^3}, \&c. \text{ ad infinitum}$.

Now since $\frac{1}{r-1} = \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c.$ as above.

Th. $\frac{1}{r^n} \times \frac{1}{r-1} = \frac{1}{r^n} \times \frac{1}{r} + \frac{1}{r^n} \times \frac{1}{r^2} + \frac{1}{r^n} \times \frac{1}{r^3}, \&c. \text{ ad infinitum}$.

Whence the sum of n terms of the given series will be

$$\frac{1}{r-1} - \frac{1}{r^n} \times \frac{1}{r-1}.$$

If the above series be applied to computations of compound interest, that is, if r be the amount of 1 £. in 1 year; then $\frac{1}{r^n}$ will be the present worth of 1 £. due at

the end of n years. Let $\frac{1}{r^n}$, therefore $= p$,

Then $\frac{1}{r-1} - \frac{1}{r^n} \times \frac{1}{r-1}$ will become $\frac{1}{r-1} - \frac{p}{r-1}$,

Therefore $\frac{1-p}{r-1}$ will be the value required. 1

SCHOLIUM. The given series, continued to n terms, viz. $\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} (n^*)$, is equal to the present worth of an annuity to continue n years certain: Now if such an annuity be denoted by A , then $A = \frac{1-p}{r-1}$.

* When the Letter (n) is placed, as above, after the initial terms of a series, it is designed to denote that the series is to be continued to n terms, and no more.

QUESTION XVI.

The value of n terms of the series $\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}$, &c. is required.

SOLU-

SOLUTION.

If $\frac{z}{r-1} = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}, \text{ \&c. ad infinitum.}$

Then $z = r.$

For if $\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4}, \text{ \&c. be multiplied}$

By

$$r - 2r + 1$$

$$r + 2 + \frac{3}{r} + \frac{4}{rr}, \text{ \&c.}$$

$$- 2 - \frac{4}{r} - \frac{6}{rr}, \text{ \&c.}$$

$$+ \frac{1}{r} + \frac{2}{rr}, \text{ \&c.}$$

The product is $= r + 0 + 0 + 0, \text{ \&c.}$

Therefore $\frac{r}{r-1}$ is the value of the series infinitely continued.

Now the terms, which follow the n th term, in the given series are $\frac{n+1}{r^{n+1}} + \frac{n+2}{r^{n+2}} + \frac{n+3}{r^{n+3}}, \text{ \&c. ad infin.}$

Or $\frac{1}{r^n} \times \frac{n+1}{r} + \frac{n+2}{r^2} + \frac{n+3}{r^3}, \text{ \&c. ad infin.}$

Which

Which series may be divided into two others, viz.

$$\frac{1}{r^n} \times \frac{n}{r} + \frac{n}{r^2} + \frac{n}{r^3}, \&c. = \frac{n}{r^n} \times \frac{1}{r-1} \text{ by quest. 15 ; And}$$

$$\frac{1}{r^n} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3}, \&c. = \frac{1}{r^n} \times \frac{r}{r-1}, \text{ by the above.}$$

Now if the sum of these two series be subtracted from the former, the remainder will be the value required,

$$\text{viz. } \frac{r}{r-1} - \frac{n}{r^n} \times \frac{1}{r-1} - \frac{1}{r^n} \times \frac{r}{r-1} ; \text{ in which}$$

$$\text{if } \frac{1}{r^n} = p, \text{ it will be } \frac{r}{r-1} - \frac{rp}{r-1} - \frac{np}{r-1} ;$$

$$\text{Or } \frac{1-p \times r}{r-1} - \frac{np}{r-1}.$$

QUESTION XVII.

The value of n terms of the series, $\frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4} + \frac{25}{r^5}, \&c.$ is required ?

SOLUTION.

$$\text{If } \frac{z}{r-1} = \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4} + \frac{25}{r^5}, \&c.$$

$$\text{Then } z = (rr + r) \overline{r+1} \times r.$$

For

For if $\frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4} + \frac{25}{r^5}, \&c.$ be multiplied

By $r^3 - 3r^2 + 3r - 1$

$$rr + 4r + 9 + \frac{16}{r} + \frac{25}{rr}, \&c.$$

$$- 3r - 12 - \frac{27}{r} - \frac{48}{rr}, \&c.$$

$$+ 3 + \frac{12}{r} + \frac{27}{rr}, \&c.$$

$$- \frac{1}{r} - \frac{4}{rr}, \&c.$$

The product is $\left. \begin{array}{l} \end{array} \right\} rr + r + 0 + 0 + 0, \&c.$

Therefore the given series infinitely continued is worth

$$\frac{r+1 \times r}{r-1^3}$$

Now the terms, which follow the n th term, in the given series are $\frac{n+1^2}{r^{n+1}} + \frac{n+2^2}{r^{n+2}} + \frac{n+3^2}{r^{n+2}}, \&c.$

$$\&c. \frac{nn+2n+1}{r^{n+1}} + \frac{nn+4n+4}{r^{n+2}} + \frac{nn+6n+9}{r^{n+2}}, \&c.$$

Which series may be divided into three others, viz.

$$\frac{nn}{r^n} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{nn}{r^n} \times \frac{1}{r-1} \quad \text{By quest. 15.}$$

$$\frac{2n}{r^n} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3}, \&c. = \frac{2n}{r^n} \times \frac{r}{r-1^2} \quad \text{By quest. 16.}$$

$$\frac{1}{r^n} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3}, \&c. = \frac{1}{r^n} \times \frac{r+1 \cdot r}{r-1^3} \quad \text{By above:}$$

In which three series, writing p for $\frac{1}{r^n}$ and subtracting their sum from the above found value of the whole series, there remains $\frac{1-p \times r + 1 \times r}{r-1^3} - \frac{2nrp}{r-1^2} - \frac{nnp}{r-1}$ the value of n terms thereof, which was required.

Q U E S T I O N X V I I I .

The value of n terms of the series, $\frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} + \frac{125}{r^5}$, &c. is required?

S O L U T I O N .

If $\frac{x}{r-1^4} = \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} + \frac{125}{r^5}$, &c. *ad infin.*

Then $x = r^3 + 4r^2 + r$, as will appear, if the multiplication be performed, after the manner of the three last questions.

But $\overline{r+2^2} = rr + 4r + 4$,

And $\overline{r+2 \times r} = r^3 + 4rr + 4r$;

Th. $\overline{r+2 \times r} - x = 3r$,

And $x = \overline{r+2^2} \times r - 3r$.

Therefore the series, infinitely continued, will be in va-

lue $\frac{\overline{r \cdot r + 2 - 3r}}{r-1^4}$

Now

Now the terms, which follow the n th term, will be

$$\frac{n+1^3}{r^{n+1}} + \frac{n+2^3}{r^{n+2}} + \frac{n+3^3}{r^{n+3}}, \&c.$$

$$\text{Or } \frac{n^3+3n^2+3n+1}{r^{n+1}} + \frac{n^3+6n^2+12n+8}{r^{n+2}} + \frac{n^3+9n^2+27n+27}{r^{n+3}}, \&c.$$

Which series may be divided into 4 others, *viz.*

$$\frac{n^3}{r^n} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{n^3}{r^n} \times \frac{1}{r-1} \text{ By quest. 15.}$$

$$\frac{3n^2}{r^n} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3}, \&c. = \frac{3n^2}{r^n} \times \frac{r}{r-1} \text{ By quest. 16.}$$

$$\frac{3n}{r^n} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3}, \&c. = \frac{3n}{r^n} \times \frac{r \cdot r + 1}{r-1} \text{ By quest. 17.}$$

$$\frac{1}{r^n} \times \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3}, \&c. = \frac{1}{r^n} \times \frac{r \cdot r + 2 - 3r}{r-1} \text{ as above.}$$

Whence if p be written for $\frac{1}{r^n}$ in the value of those 4 series, and their sum be subtracted from the before found value of the given series, infinitely continued, the remainder will be,

$$\frac{1-p \times r \cdot 1+2-3r}{r-1^4} - \frac{r+1 \times 3nrp}{r-1^3} - \frac{3nrp}{r-1^2} - \frac{n^3 p}{r-1},$$

which is the value of n terms as was required.

QUES.

QUESTION XIX.

The value of n terms of the series $\frac{1}{r} + \frac{16}{r^2} + \frac{81}{r^3} + \frac{256}{r^4} + \frac{625}{r^5}$, is required?

SOLUTION.

If $\frac{z}{r-1} = \frac{1}{r} + \frac{16}{r^2} + \frac{81}{r^3} + \frac{256}{r^4} + \frac{625}{r^5}$, &c. *ad infini*

Then $z = r^4 + 11r^3 + 11r^2 + r$, as will appear by multiplying both sides of the equation by $r-1$.

Therefore the series infinitely continued will be $= \frac{r^4 + 11r^3 + 11r^2 + r}{r-1}$.

Now the terms, which follow the n th term, will be

$$\frac{n+1^4}{r^{n+1}} + \frac{n+2^4}{r^{n+2}} + \frac{n+3^4}{r^{n+3}}, \&c.$$

Or

$$\begin{aligned} & \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{r^{n+1}} \\ & + \frac{n^4 + 8n^3 + 24n^2 + 32n + 16}{r^{n+2}} \\ & + \frac{n^4 + 12n^3 + 54n^2 + 108n + 81}{r^{n+3}} \end{aligned}$$

Which

Which series may be divided into the five following, viz.

$$\frac{n^4}{r^n} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}, \&c. = \frac{n^4}{r^n} \times \frac{1}{r-1} \text{ By quest. 15.}$$

$$\frac{4n^3}{r^n} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3}, \&c. = \frac{4n^3}{r^n} \times \frac{r}{r-1} \text{ By quest. 16.}$$

$$\frac{6n^2}{r^n} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3}, \&c. = \frac{6n^2}{r^n} \times \frac{r+1 \times r}{r-1} \text{ By quest. 17.}$$

$$\frac{4n}{r^n} \times \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3}, \&c. = \frac{4n}{r^n} \times \frac{r^3+4r^2+r}{r-1} \text{ 18.}$$

$$\frac{1}{r^n} \times \frac{1}{r} + \frac{16}{r^2} + \frac{81}{r^3}, \&c. = \frac{1}{r^n} \times \frac{r^4+11r^3+11r^2+r}{r-1}$$

Whence, if we write p for $\frac{1}{r^n}$, and subtract these 5

series from the value of the given series, infinitely continued, there will remain

$$\frac{1}{1-p} \times \frac{r^4+11r^3+11r^2+r}{r-1} - 4n p \times \frac{r^3+4r^2+r}{r-1}$$

$$\left(-6n^2 p \times \frac{r^2+r}{r-1} - 4n^3 p \times \frac{r}{r-1} - n^4 p \times \frac{1}{r-1} \right).$$

the value of n terms of the required series.

COROL. If we put $\frac{1}{r-1} = P$; $\frac{r}{r-1} = Q$; $\frac{r^2+r}{r-1}$

$$= R; \frac{r^3+4r^2+r}{r-1} = S; \frac{r^4+11r^3+11r^2+r}{r-1} = T, \&c.$$

then may the sum of n terms of the series $\frac{1^n}{r} + \frac{2^n}{r^2}$

+

$$\begin{aligned}
 & + \frac{3^m}{r^3} + \frac{4^m}{r^4} + \frac{5^m}{r^5}, \text{ \&c. be expressed by } -n^m p \times P \\
 & - m n^{m-1} p \times Q - \frac{m \cdot m-1}{1 \cdot 2} n^{m-2} p \times R - \\
 & \frac{m \cdot m-1 \cdot m-2}{1 \cdot 2 \cdot 3} n^{m-3} p \times S, \text{ \&c. } + \frac{1}{1-p} \times Z.
 \end{aligned}$$

Where Z signifies the sum of the series $\frac{1^m}{r} + \frac{2^m}{r^2} + \frac{3^m}{r^3}, \text{ \&c. ad infinitum.}$

It only remains, therefore, in the succeeding questions to find the value of $V, W, X, Y, \text{ \&c.}$ the sums of the following series when infinitely continued.

QUESTION XX.

What is V the value of $\frac{1}{r} + \frac{32}{rr} + \frac{243}{r^3} + \frac{1024}{r^4}, \text{ \&c. ad infinitum.}$

E

SOLU.

SOLUTION.

Now to proceed as before,

$$\begin{array}{r}
 \text{Multiply } \frac{1}{r} + \frac{32}{rr} + \frac{243}{r^3} + \frac{1024}{r^4} + \frac{3125}{r^5}, \text{ \&c.} \\
 \text{by } r^5 - 6r^4 + 15r^3 - 20r^2 + 15r^2, \text{ \&c.} \\
 \hline
 r^5 + 32r^4 + 243r^3 + 1024r^2 + 3125r, \text{ \&c.} \\
 - 6r^4 - 192r^3 - 1458r^2 - 6144r, \text{ \&c.} \\
 + 15r^3 + 480r^2 + 3645r, \text{ \&c.} \\
 - 20r^2 - 640r, \text{ \&c.} \\
 + 15r, \text{ \&c.} \\
 \hline
 \end{array}$$

The prod. will be $r^5 + 26r^4 + 66r^3 + 26r^2 + r, \text{ \&c.}$

$$\text{Therefore } V = \frac{r^5 + 26r^4 + 66r^3 + 26r^2 + r}{r - 1}$$

C O R O L.

From the foregoing solutions, we may conclude,

1st. That the number of terms, in the numerator of the fraction, which exhibits the value of these kind of infinite series, will be always equal to m , the index of that power to which the numbers, 1, 2, 3, &c. are raised in the numerators of the terms constituting the series.

2^d. That, after the numeral coefficients of half those terms are found, the following coefficients will be the same with the former, but in a reversed order; after the manner of the unciæ of a binomial.

3^d. That the denominator of the said fraction will always be $r - 1$ raised to a power $(m+1)$ whose index is greater, by unity, than the index of that power to which the numbers, 1, 2, 3, &c. are raised in the numerators of the terms constituting the series.

Also

Also, from the nature of the operation by which the numerator of the said fraction is found, it will appear :

4. That the coefficient of the first term thereof will be unity.

5. The coefficient of the second term, $2^m - \frac{m+1}{1}$.

6. That of the third, $3^m - 2^m \times \frac{m+1}{1} + \frac{m+1 \cdot m}{1 \cdot 2}$.

7. That of the fourth, $4^m - 3^m \times \frac{m+1}{1} + 2^m \times \frac{m+1 \cdot m}{1 \cdot 2} - \frac{m+1 \cdot m \cdot m-1}{1 \cdot 2 \cdot 3}$.

And therefore Z , the value of the series $\frac{1^m}{r} + \frac{2^m}{r^2} + \frac{3^m}{r^3} + \frac{4^m}{r^4}$, &c. *ad infinitum*, will be

$$\frac{1}{1-r} \left\{ \begin{aligned} & r^m + r^{m-1} \times 2 - \frac{m \cdot m+1}{1} + \\ & r^{m-2} \times 3^m - 2^m \times \frac{m+1}{1} + \frac{m+1 \times m}{1 \times 2} + \\ & r^{m-3} \left\{ 4^m - 3^m \times \frac{m+1}{1} + 2^m \times \frac{m+1 \times m}{1 \cdot 2} - \frac{m+1 \times m \times m-1}{1 \times 2 \times 3} \right\} (m) \end{aligned} \right.$$

Thus are we furnished with the means of finding the numbers W , X , Y , &c.

First, for W ; Where $m=6$: Then $2^6=64$; $\frac{m+1}{1}=7$;

And $64-7=57$; $3^6=729$; $64 \times 7=448$; $7 \times \frac{6 \cdot 5}{1 \cdot 2}=21$;

And $729-448+21=302$:

$$\text{Th. } W = \frac{r^6 + 57r^5 + 302r^4 + 302r^3 + 57r^2 + r}{r-1^7}.$$

Secondly, to find X , $m=7$: Then $2^7=128$; $m+1=8$;
 And $128-8=120$: $3^7=2187$; $128 \times 8=1024$; $8 \times \frac{7}{2}=28$,
 And $2187-1024+28=1191$: $4^7=16384$; 2187×8
 $=17496$; $128 \times 28=3584$; $28 \times \frac{6}{2}=56$; And $16384 -$
 $17496+3584-56=2416$.

Th. X is

$$= \frac{r^7 + 120r^6 + 1191r^5 + 2416r^4 + 1191r^3 + 120r^2 + r}{r-1^8}.$$

Thirdly, to find Y , $m=8$: Then $2^8=256$; $m+1=9$;
 And $256-9=247$: $3^8=6561$; $256 \times 9=2304$; $9 \times \frac{8}{2}=36$;
 And $6561-2304+36=4293$; $4^8=65536$; $6561 \times 9 =$
 $59049+9216-84=15619$.

Th. $Y =$

$$\frac{r^8 + 247r^7 + 4293r^6 + 15619r^5 + 15619r^4 + 4293r^3 + 247r^2 + r}{r-1^9}.$$

As the numeral values of these expressions, at the different rates of interest, and their logarithms, will be of great service in the speedy resolution of questions, relating to the values of combined lives, the following table is inserted.

Rate

REPOSITORY.

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Rate per Ct.	P	Q	R	S	T	V
3	$\frac{100}{3}$	$\frac{10300}{9}$	77440,741	7859663	1053596960	179912043600
$3\frac{1}{2}$	$\frac{200}{7}$	$\frac{41400}{49}$	49124,782	4283960	498114240	72397376300
4	$\frac{25}{200}$	$\frac{650}{41800}$	33150	2535650	258603154	32967678466
$4\frac{1}{2}$	$\frac{9}{20}$	$\frac{81}{420}$	23451,575	1598358	145249496	16499293600
5	$\frac{50}{3}$	$\frac{2650}{9}$	17220	1058820	8686019	8895852030
6			10109,259	520480	35729490	3065913930
3	$\frac{1,5228787}{1,4559320}$	$\frac{3,058946}{2,9268043}$	$\frac{4,8889693}{4,6913007}$	$\frac{6,8954039}{6,6318456}$	$\frac{9,0267769}{8,6973292}$	$\frac{11,2550600}{10,8597228}$
$3\frac{1}{2}$	$\frac{1,3979400}{1,3467875}$	$\frac{2,8129134}{2,7120913}$	$\frac{4,5204835}{4,3701721}$	$\frac{6,4040893}{6,2036741}$	$\frac{8,4126338}{8,1621147}$	$\frac{10,5180883}{10,2174655}$
4	$\frac{1,3040300}{1,2218487}$	$\frac{2,6232493}{2,4690033}$	$\frac{4,2360334}{4,0047192}$	$\frac{6,0248221}{5,7164935}$	$\frac{7,9385499}{7,5530266}$	$\frac{9,9491876}{9,4865597}$

E 3

QUES.

QUESTION XXI.

If S be the sum of n terms of the arithmetical progression $a, a-d, a-2d, a-3d, \&c.$ whose common difference d is known; and if Z be the sum of n terms of another arithmetical progression $b, b-d, b-2d, b-3d, \&c.$ whose common difference δ is also known; and if there be farther given SZ the product of the said sums, it is thence required to find the sum of n terms of the following series, *viz.* $ab + \overline{a-d} \times \overline{b-d} + \overline{a-2d} \times \overline{b-2d}, \&c.$ which is formed by multiplying the several corresponding terms of each arithmetical progression together?

SOLUTION.

$$\text{By quest. 3. part 2. vol. I. } S = na - \frac{n \cdot n - 1}{1 \cdot 2} d.$$

$$\text{And } Z = nb - \frac{n \cdot n - 1}{1 \cdot 2} \delta;$$

$$\text{Th. } SZ = nna\overline{b} - \frac{nn \cdot n - 1}{1 \cdot 2} \overline{bd} - \frac{nn \cdot n - 1}{1 \cdot 2} a\overline{\delta} + \frac{nn \cdot n - 1}{2 \cdot 2} \overline{d\delta}.$$

which expresses the given product.

The terms of the series $ab + \overline{a-d} \times \overline{b-d} + \overline{a-2d} \times \overline{b-2d}, \&c.$ may be represented as follows, *viz.*

$$\overline{ab} = ab,$$

$$\overline{a-d} \times \overline{b-d} = ab - bd - ad + d\delta,$$

$$\overline{a-2d} \times \overline{b-2d} = ab - 2bd - 2ad + 4d\delta,$$

$$\overline{a-3d} \times \overline{b-3d} = ab - 3bd - 3ad + 9d\delta;$$

$$\&c. \quad \&c. \quad \&c. \quad \&c. \quad \&c.$$

The

The sum of which may be divided into the four following series, *viz.*

$$\begin{aligned}
 ab \times \overline{1+1+1+1} (n) &= nab, \\
 bd \times \overline{0+1+2+3} (n) &= \frac{n \cdot n-1}{1 \cdot 2} bd, \text{ See qn. 1.} \\
 ad \times \overline{0+1+2+3} (n) &= \frac{n \cdot n-1}{1 \cdot 2} ad, \\
 dd \times \overline{0+1+4+9} (n) &= \frac{n \cdot n-1 \cdot 2n-1}{1 \cdot 2 \cdot 3} dd; 33
 \end{aligned}
 \left. \vphantom{\begin{aligned} ab \times \overline{1+1+1+1} (n) \\ bd \times \overline{0+1+2+3} (n) \\ ad \times \overline{0+1+2+3} (n) \\ dd \times \overline{0+1+4+9} (n) \end{aligned}} \right\} \text{Part 2. Vol. 1.}$$

Therefore putting Σ = the sum of n terms of the said series ab , $a-d \times b-d + a-2d \times b-2d$, &c.

$$\Sigma = nab - \frac{n \cdot n-1}{1 \cdot 2} bd - \frac{n \cdot n-1}{1 \cdot 2} ad + \frac{n \cdot n-1 \cdot 2n-1}{1 \cdot 2 \cdot 3} dd;$$

Now if the above found value of SZ be divided by n

$$\frac{SZ}{n} = nab - \frac{n \cdot n-1}{1 \cdot 2} bd - \frac{n \cdot n-1}{1 \cdot 2} ad + \frac{n \cdot n-1 \cdot 2n-1}{1 \cdot 2 \cdot 2} dd;$$

Whence by subtraction,

$$\Sigma - \frac{SZ}{n} = \frac{n \cdot n-1 \cdot 2n-1}{1 \cdot 2 \cdot 3} dd - \frac{n \cdot n-1 \cdot n-1}{1 \cdot 2 \cdot 2} dd, \text{ Or}$$

$$\Sigma - \frac{SZ}{n} = \frac{n \cdot n-1}{1 \cdot 2} dd \times \frac{2n-1}{3} - \frac{n-1}{2};$$

$$\text{But } \frac{n+1}{6} = \frac{2n-1}{3} - \frac{n-1}{2}; \text{ Therefore}$$

$$\Sigma - \frac{SZ}{n} = \left(\frac{n \cdot n-1}{1 \cdot 2} dd \times \frac{n+1}{6} \right) = \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} dd.$$

Th. $\Sigma = \frac{SZ}{n} + \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} dd$ is the value of n terms of the required series.

QUESTION XXII.

$$\text{If } S = a + \overline{a-d} + \overline{a-2d} (n),$$

$$Z = b + \overline{b-\delta} + \overline{b-2\delta} (n),$$

$$W = c + \overline{c-\Delta} + \overline{c-2\Delta} (n),$$

$$\text{And } \Sigma = abc + \overline{a-d} \times \overline{b-\delta} \times \overline{c-\Delta} + \\ (\overline{a-2d} \times \overline{b-2\delta} \times \overline{c-2\Delta}) (n);$$

Then having given SZW , (the product of the three sums of the different arithmetical progressions) and what else may be necessary given, Σ is required?

SOLUTION.

$$\text{Since } S = na - \frac{n \cdot n-1}{1 \cdot 2} d,$$

$$Z = nb - \frac{n \cdot n-1}{1 \cdot 2} \delta,$$

$$\text{And } W = nc - \frac{n \cdot n-1}{1 \cdot 2} \Delta;$$

It will follow, that,

$$SZW = \left\{ \begin{array}{l} n^3 abc - \frac{n^3 \cdot n-1}{2} + ab\Delta + ac\delta + bcd + \\ \frac{n^3 \cdot n-1^2}{2 \cdot 2} \times a\delta\Delta + bd\Delta + cd\delta - \frac{n^3 \cdot n-1^3}{2 \cdot 4} d\delta\Delta \end{array} \right.$$

As will appear by the continual multiplication of the several values of S , Z and W ; Therefore

SZW

$$\frac{SZW}{nn} = \left\{ \begin{aligned} nabc - \frac{n \cdot n-1}{1 \cdot 2} \times \overline{a\delta \Delta + a\epsilon\delta + b \cdot d} + \\ \frac{n \cdot n-1^2}{2 \cdot 2} \times \overline{a\delta \Delta + bd\Delta + cd\delta} - \frac{n \cdot n-1^3}{2 \cdot 4} d\delta \Delta; \end{aligned} \right.$$

And, by proceeding as in the last question, it will appear that n terms of the series $abc + \overline{a-d} \times \overline{b-\delta} \times \overline{c-\Delta} + \overline{a-2d} \times \overline{b-2\delta} \times \overline{c-2\Delta} + \overline{a-3d} \times \overline{b-3\delta} \times \overline{c-3\Delta}$, &c. That is

$$\Sigma = \left\{ \begin{aligned} nabc - \frac{n \cdot n-1}{1 \cdot 2} \times \overline{ab\Delta + a \cdot \delta + bcd} + \\ \frac{n \cdot n-1 \cdot 2n-1}{1 \cdot 2 \cdot 3} \times \overline{a\delta \Delta + bd\Delta + cd\delta} - \frac{n^2 \cdot n-1^2}{2 \cdot 2} d\delta \Delta; \end{aligned} \right.$$

$$\text{Th. } \Sigma - \frac{SZW}{nn} = \left\{ \begin{aligned} \frac{2n-1}{3} - \frac{n-1}{2} \times \frac{n \cdot n-1}{1 \cdot 2} \times \overline{a\delta \Delta + bd\Delta + cd\delta} \\ - \frac{n \cdot n-1}{2} - \frac{n-1^2}{4} \times \frac{n \cdot n-1}{1 \cdot 2} d\delta \Delta; \end{aligned} \right.$$

$$\text{Now } \frac{n+1}{6} = \frac{2n-1}{3} - \frac{n-1}{2},$$

$$\text{And } \frac{n+1 \cdot n-1}{2 \cdot 2} = \frac{n \cdot n-1}{1 \cdot 2} - \frac{n-1^2}{4};$$

$$\text{Th. } \Sigma - \frac{SZW}{nn} = \left\{ \begin{aligned} \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \overline{a\delta \Delta + bd\Delta + cd\delta} \\ - \frac{n+1 \cdot n \cdot n-1^2}{2 \cdot 2 \cdot 2} d\delta \Delta, \end{aligned} \right.$$

$$\text{And } \Sigma = \frac{SZW}{nn} + \left\{ \begin{aligned} \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \overline{a\delta \Delta + bd\Delta + cd\delta} \\ - \frac{n+1 \cdot n \cdot n-1^2}{2 \cdot 2 \cdot 2} d\delta \Delta. \end{aligned} \right.$$

QUESTION XXIII.

Suppose a round piece of metal, equably formed, having two opposite faces, one white, the other black, be thrown up, in order to see which of those faces will be uppermost, after the metal is fallen to the ground; when, if the white face appears uppermost, a person is to be entitled to 5 *l.* or any other sum of money: It is required to determine, before the event, what chance, or probability, that person hath of receiving the 5 *l.* and what sum he may reasonably expect should be paid to him, in consideration of his resigning his chance to another?

SOLUTION.

Since, by supposition, there is nothing in the form of the metal that can incline it to shew one face rather than the other; and since it must necessarily shew, either the white, or the black face, it will follow, that there is an equal chance for the appearance of either face; or, in other words, that there is one chance, out of two, for the appearance of the white face; and, consequently, the probability thereof may be express'd by the fraction $\frac{1}{2}$: If, therefore, any other person should be willing to purchase this chance, the proprietor may reasonably expect $\frac{1}{2}$ of the 5 *l.* in consideration of his resigning thereof.

QUES-

QUESTION XXIV.

Suppose there are three cards, each of different suits, viz, one heart, one diamond; and one club, laid on a table with their faces downward; out of which, if a person at one trial takes the heart, he is to be entitled to five pounds, or any other sum of money: It is required to determine, before the event, what chance, or probability, he hath of winning and missing the said five pounds, and what sum he may reasonably expect to be paid to him, in consideration of his resigning the chance to another?

SOLUTION.

Since there is nothing on the outside, whereby the person choosing can judge or determine whether of the three cards, exposed to his view, is the heart; and since he is to have but one choice, it will follow, that he hath but one chance in three, for obtaining the five pounds, and that the probability thereof may be expressed by the fraction $\frac{1}{3}$: Again, since there will be two cards remaining after he has made his choice, either of which may be the heart; therefore there are two chances out of three, that he will miss it, and the probability thereof may be expressed by the fraction $\frac{2}{3}$: Lastly, he may reasonably expect $\frac{1}{3}$ of the five pounds, as a consideration for transferring his chance to another.

QUESTION XXV.

Suppose that there are five counters, four whereof are black, and one white; out of which (being mixed together) a person blindfolded is to draw one, and is to be entitled to five pounds, or any other sum, if he happens to draw that which is white : It is required to determine, before the event, what chance, or probability, he has of winning and missing the said five pounds, and what sum he may reasonably expect to be paid to him, for resigning his chance to another.

SOLUTION.

Since the person, who is to draw the counter, is supposed to be deprived of his sight, he cannot form any judgment, which of the five counters is the white one, or prize, and since he is confined to the taking, only, one of them, it will follow, that he hath only one chance in five, for obtaining the five pounds, and that the probability thereof may be expressed by the fraction $\frac{1}{5}$: Again, since there will be four counters remaining, after he hath drawn out one, either of which may be the prize ; therefore there are four chances out of the five for his missing it, and the probability thereof may be expressed by the fraction $\frac{4}{5}$: Lastly he will be entitled to $\frac{1}{5}$ of the five pounds, if he transfers his chance to another.

QUES-

QUESTION XXVI.

Suppose there are five counters, three whereof are black, and the other two white, out of which, when mixed together, a person blindfolded is to draw one, and is to be entitled to five pounds, or any other sum, if he happens to draw either of the white ones: It is required to determine, before the event, what chance, or probability, he has of winning and missing the said five pounds, and what sum he may reasonably expect to be paid to him, for resigning his chance to another?

SOLUTION.

As the person that draws cannot know which three of the five counters are black, and which two are white; and as he is to take but one of them; it is plain that he hath only two out of the five chances for taking a white counter, and the other three for taking a black one; and consequently the probability of winning may be expressed by the fraction $\frac{2}{5}$, and that of missing by $\frac{3}{5}$: Lastly, he ought to receive $\frac{2}{5}$ of the five pounds, if he parts with his chance to another.

SCHOLIUM. What has been said, in these four questions, concerning cards, counters, &c. may be very well conceived to extend to any other things, which are the objects of chance: For instance, if at the conclusion of the drawing of a Lottery, there should remain in one wheel five tickets or numbers only, and in the other wheel two equal prizes, and three blanks: Then the possessor of one of those tickets, would be exactly in the state of the person mentioned in the last question.

COROL.

COROLLARIES.

Since, when the number of blanks is	$\frac{1}{3}$	And the number of prizes	$\frac{1}{1}$	And, consequent- ly, the number of tickets	$\frac{3}{1}$	The probability of having a prize with one ticket is	$\frac{1}{3}$	And the probabi- lity of a blank	$\frac{2}{3}$	The sum of which two probabilities is unity.	$\frac{1}{3} + \frac{2}{3}$
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Therefore, *first*, when in any lottery, the number of blanks is m , and the number of prizes n , then the probability of having one prize, with one ticket, will be $\frac{n}{m+n}$; and the probability of having a blank $\frac{m}{m+n}$:

Or, in other words, *the probability of the happening of any event, resulting from chance, may be expressed by a fraction; whose numerator is the number of chances for the events taking place; and the denominator the number of all the chances, whereby it may both happen and fail. And the probability of such an event's failing may be expressed by a fraction, whose numerator is the number of chances for the not happening thereof, and the denominator the same with that of the former fraction.*

Secondly, Since the sum of the two fractions, representing the probabilities of the happening and failing of an event, is unity; therefore, the one of them being given, the other may be found by subtraction.

Thirdly, The expectation (that is, the sum which the person, who has a chance for the happening of an event) is entitled to, if he surrenders that chance to another, is always the product of the fraction, representing the probability, multiplied into the sum expected; and if the

the sum expected be denoted by unity, then the expectation will be denoted by the probability itself. Examples of which will frequently occur hereafter.

QUESTION XXVII.

A person, playing with a single die, offers to lay a wager, that he will throw an ace, each time, for two successive throws: What probability has he of succeeding therein?

SOLUTION.

Suppose the wager to be 36*l.* and that, on throwing, the first time, an ace did come up: Then because there are six faces on the die, one of which (only) is favourable to him, his expectation on the second throw will be ($\frac{1}{6}$ of 36*l.* or) 6*l.* Now we may conceive the first event, *viz.* the throwing an ace the first throw, as the condition of obtaining this 6*l.* but the probability of this event being also $\frac{1}{6}$, the expectation, before the first throw, must necessarily be ($\frac{1}{6}$ of 6*l.* or) 1*l.* which being the $\left(\frac{1}{6} \times \frac{1}{6} =\right)$

$\frac{1}{36}$ part of the sum wagered, if that wager had been 1*l.* the expectation, and, consequently, the probability of winning thereof, would necessarily have been $\frac{1}{36}$. And therefore, the probability of his losing the wager will be $\left(1 - \frac{1}{36} =\right) \frac{35}{36}$; whence it appears, that no person should lay that wager, unless 35 to 1 in value, be laid against him.

QUES-

QUESTION XXVIII.

A person offers to lay a wager of 1*l*. that out of a purse containing $m+n$ counters, whereof m are black and n white, he will, blindfolded, at the first trial, draw a white counter ; and also that, out of another purse, containing $M+N$ counters, whereof M are black, and N white, he will also, blindfolded, at the first trial, draw a white counter ; and that, if he fails in either trial, his wager shall be lost : It is required to determine the probability he has of succeeding therein.

SOLUTION.

Now if, as in the last question, we for the present suppose he has already succeeded in the first trial, then it will follow, from what is foregoing, that his expectation on the second will be $\frac{N}{M+N}$: And if, as in the same question, we conceive the success of the first trial as the condition of obtaining this expectation, then the probability of so doing will be $\frac{n}{m+n}$; which multiplied into that expectation (viz. $\frac{N}{M+N}$) will give $\frac{n}{m+n} \times \frac{N}{M+N}$ for the probability required.

Therefore, the probability of the happening of two *independent* events, will be equal to the product of the probabilities of their happening separately.

COROL

C O R O L.

If the two events are of the same kind, then the probability will be $\frac{n \cdot n}{m+n^2}$

Q U E S T I O N XXIX.

A person offers to lay a wager of 1 l. that out of a purse containing $m+n$ counters, whereof m are black and n white, he will, blindfolded, at the first trial, draw a black counter; and also that, out of another purse containing $M+N$ counters, whereof M are black and N white, he will also, blindfolded, draw, at the first trial, a white counter; and that if either experiment fails he will lose his wager: It is required to determine the probability he has of succeeding therein?

S O L U T I O N.

If the argument of the last question be followed the answer will be $\frac{m}{m+n} \times \frac{N}{M+N}$; but as this question may be taken in a different light from the former, viz. that he is to fail of drawing a white counter at the first trial, and to succeed therein in the second; and then the probability of the failing of the first trial $\left(1 - \frac{n}{m+n}\right)$ being multiplied by $\left(\frac{N}{M+N}\right)$ the probability of succeeding in the second; the product (viz. $1 - \frac{n}{m+n} \times \frac{N}{M+N}$) which will be the probability required.

And

And in like manner, the probability of failing in both experiments will be the product of the probabilities of their

separately failing, viz. $1 - \frac{n}{m+n} \times 1 - \frac{N}{M+N}$.

C O R O L L.

If the events, above spoken of, be exactly of the same kind, then the probability of failing in the first trial, and

succeeding in the second, will be $1 - \frac{n}{m+n} \times \frac{n}{m+n}$.

And the probability of failing in both $1 - \frac{n}{m+n} \times$

$$1 - \frac{n}{m+n}.$$

Q U E S T I O N XXX.

What is the probability of throwing with a single die, one ace, or more, in two throws; that is, either at the first or second throw, or at both?

S O L U T I O N.

Let 6 (the number of the faces of the die) = n

Then, the probability of throwing the required face, the first throw, will be $\frac{1}{n}$; and the probability of missing

$$\text{it } \left(1 - \frac{1}{n}\right) \frac{n-1}{n},$$

Suppose

Suppose it missed at the first throw; and then the probability of throwing it, the second time, will be also $\frac{1}{n}$; but if this probability be valued before the first throw

is made, it must be connected with $\frac{n-1}{n}$ the probability of missing it then, (for if it be thrown at first, the condition of the question is performed, and there is no occasion to throw a second time): And the probability of missing it the first time, and throwing it the second, will be $\frac{n-1}{n} \times \frac{1}{n} = \frac{n-1}{nn}$.

Therefore the probability of throwing the required face, either at the first or second throw, will be $\left(\frac{1}{n} + \frac{n-1}{nn} = \frac{n+n-1}{nn} = \frac{2n-1}{nn}\right) \frac{n-1}{nn}$; In this

case $\left(\frac{36-25}{36} = \frac{11}{36}\right)$.

This question may be more readily answered by finding the probability of missing an ace twice, and subtracting that probability from unity; for then the remainder will be the probability of throwing one ace at least in two throws.

Thus, the probability of missing an ace the first throw is $\frac{n-1}{n}$, and the probability of missing it the second throw is also $\frac{n-1}{n}$; therefore the probability of missing

an ace twice is $\left(\frac{n-1}{n} \times \frac{n-1}{n} = \frac{n-1}{nn}\right)$.

And

And $\left(1 - \frac{n-1}{nn} = \frac{nn-n-1}{nn}\right)$ will be the probability of throwing one ace at least in two throws.

QUESTION XXXI.

What is the probability of throwing an ace (or any one of the faces of the die) in three throws, that is, either at the first, second, or third throw?

SOLUTION.

The probability of throwing the required face in two throws is $\frac{nn-n-1}{nn}$, and the probability of missing it the two first throws is $\frac{n-1}{nn}$ by question 30.

Therefore the probability of missing it the two first throws, and throwing it the third, will be $\left(\frac{n-1}{nn} \times \frac{1}{n}\right) = \frac{n-1}{n^3}$.

Therefore, the probability of throwing the required face, either at the first, second, or third throw, will be $\frac{nn-n-1}{nn} + \frac{n-1}{n^3}$.

$$\text{Or } \frac{n^3-n \times n-1}{n^3} + \frac{n-1}{n^3} :$$

$$\text{But } n \times n-1 = n-1 = n-1^3,$$

For

$$\text{For } \begin{cases} \text{From } n^3 - 2nn + n, \\ \text{Take } nn - 2n + 1; \\ \text{Remains } n^3 - 3nn + 3n - 1. \end{cases}$$

Therefore $\frac{n^3 - n - 1}{n^3}$ will be the probability required.

In this case $\left(\frac{216 - 125}{216} = \right) \frac{91}{216}$.

Otherwise, take $\left(\frac{n-1}{n} \times \frac{n-1}{n} \times \frac{n-1}{n} = \right) \frac{n-1}{n^3}$, the probability of missing an ace three times successively, from unity, and the remainder $\left(1 - \frac{n-1}{n^3} = \right) \frac{n^3 - n - 1}{n^3}$ will be the probability of throwing an ace once or more, in three throws, as before.

C O R O L.

Hence, the probability of throwing one ace, or more, in m throws, may be found by subtracting $\left(\frac{n-1}{n^m}\right)$ the probability of missing an ace m times, from unity; which being done the remainder $\left(1 - \frac{n-1}{n^m} = \right) \frac{n^m - n - 1}{n^m}$ is the said probability.

Q U E S T I O N XXXII.

The probability of throwing one ace, and no more, in two throws is required?

SOLU-

SOLUTION.

From $\left(\frac{n-1}{n}\right)^2$ the probability of throwing one ace, or more, take $\frac{1}{n}$ the probability of throwing two aces, in two throws; and the remainder $\left(\frac{n-1}{n} - \frac{1}{n}\right) = \frac{n-2}{n}$ will be the probability required. In this case $\left(\frac{5-2}{6}\right) = \frac{3}{6} = \frac{1}{2}$.

QUESTION XXXIII.

What is the probability of throwing one ace, and no more, in three throws?

SOLUTION.

If an ace be thrown the first time, (the probability of which is $\frac{1}{n}$) then never an ace must be thrown in the next two Trials (the probability of which is $\left(\frac{n-1}{n}\right)^2$). And $\left(\frac{1}{n} \times \left(\frac{n-1}{n}\right)^2\right) = \frac{n-1}{n^3}$ is the probability of throwing in that order.

If

If an ace be missed the first throw (the probability of which is $\frac{n-1}{n}$) then but one ace must be thrown in the remaining two throws, (the probability of which is $\frac{n-1 \times 2}{nn}$) And $\left(\frac{n-1}{n} \times \frac{n-1 \times 2}{nn} =\right) \frac{n-1^2 \times 2}{nnn}$ is the probability of throwing in that order.

Therefore the whole probability of throwing but one ace, in three throws, will be $\left(\frac{n-1^3}{n^3} + \frac{n-1^2 \times 2}{n^3} =\right) \frac{n-1^3 \times 3}{nnn}$. In this case $\left(\frac{25 \times 3}{216} =\right) \frac{25}{72}$.

QUESTION XXXIV.

The probability of throwing one ace, and no more, in four throws, is required ?

SOLUTION.

The probability of throwing an ace the first throw, and missing it for the other three throws will be

$$\left(\frac{1}{n} \times \frac{n-1^3}{n^3} =\right) \frac{n-1^3}{n^4}.$$

And the probability of missing an ace the first time, and throwing but one in the three following throws will

$$\text{be } \left(\frac{n-1}{n} \times \frac{n-1^2 \times 3}{n^3} =\right) \frac{n-1^3 \times 3}{n^4}.$$

There-

Therefore the whole probability of throwing one ace, and no more, in 4 throws will be $\left(\frac{n-1^3}{n^4} + \frac{n-1^3 \times 3}{n^4} = \frac{n-1^3 \times 4}{n^4}\right)$. In this case $\left(\frac{125 \times 4}{1296} = \frac{125}{324}\right)$.

C O R O L.

Therefore the probability of throwing one ace, and no more, in m throws will be $\frac{n-1^{m-1} \times m}{n^m}$.

Q U E S T I O N XXXV.

What is the probability of throwing two aces, or more, in three throws?

S O L U T I O N.

If $\left(\frac{n-1^3}{n^3}\right)$ the probability of throwing never an ace in three throws, and $\left(\frac{n-1^2 \times 3}{n^3}\right)$ the probability of throwing but one ace in three throws, be both taken from unity, there will remain $\left(1 - \frac{n-1^3}{n^3} - \frac{n-1^2 \times 3}{n^3} = \frac{n^3 - n - 1^3 - n - 1^2 \times 3}{n^3}\right)$ the probability of throwing two aces, or more, in three throws.

In

In this case $\left(\frac{216-125-25 \times 3}{216} = \frac{216-200}{216} = \frac{16}{216} = \right)$

$$\frac{2}{27}$$

QUESTION XXXVI.

What is the probability of throwing two aces, or more, in four throws?

SOLUTION.

From unity, take $\left(\frac{n-1}{n^4} + \frac{n-1^3 \times 4}{n^4} \right)$ the sum of the respective probabilities of having, (in four throws) never an ace, and but 1 ace; And the remainder $\left(1 - \frac{n-1}{n^4} - \frac{n-1^3 \times 4}{n^4} \right) = \frac{n^4 - n - 1 - n-1^3 \times 4}{n^4}$ will be the probability of throwing two aces, or more, in 4 throws.

In this case $\left(\frac{1296-625-125 \times 4}{1296} = \frac{1296-1125}{1296} = \frac{171}{1296} = \right)$

$$\frac{19}{144}$$

COROLL.

Therefore the probability of throwing 2 aces, or more, in m throws will be $\frac{n^m - n - 1 - n-1^3 \times m}{n^m}$.

QUESTION XXXVII

The probability of throwing two aces, and no more, in three throws is required?

SOLUTION.

From $\left(\frac{n^3 - n - 1 - n - 1^2 \times 3}{n^3}\right)$ the probability of throw-

ing two aces, or more, in three throws, take $\frac{1}{n^3}$ the probability of throwing three aces in three throws, and the remainder $\left(\frac{n^3 - n - 1 - n - 1^2 \times 3}{n^3} - \frac{1}{n^3}\right)$ is the probability of throwing two aces, and no more, in three throws.

$$\text{But } \left(\frac{n^3 - n - 1 - n - 1^2 \times 3 - 1}{n^3} = \frac{3n - 3}{n^3} = \frac{n - 1 \times 3}{n^3}\right)$$

$$\text{For } n^3 = n^3$$

$$\text{And } \begin{cases} n - 1^3 = n^3 - 3n^2 + 3n - 1, \\ n - 1^2 \times 3 = 3n^2 - 6n + 3, \\ 1 = +1, \end{cases}$$

$$\text{Sum } \begin{array}{r} n^3 \\ - 3n^2 + 3n \\ \hline \end{array}$$

$$\text{Remainder } \begin{array}{r} 3n - 3 \\ \hline \end{array}$$

Therefore $\frac{n - 1 \times 3}{n^3}$ will be the probability of throwing two aces, and no more, in three throws.

QUES-

QUESTION XXXVIII.

The probability of throwing two aces, and no more, in 4 throws, is required?

SOLUTION.

If an ace be thrown the first throw (the probability of which is $\frac{1}{n}$) then but one ace must be thrown in the three following throws (the probability of which is $\frac{n-1^2 \times 3}{n^3}$ by quest. 33): And the probability of throwing it in this order, will be $\left(\frac{1}{n} \times \frac{n-1^2 \times 3}{n^3} =\right) \frac{n-1^2 \times 3}{n^4}$.

But if an ace be missed the first throw (the probability of which is $\frac{n-1}{n}$) then two aces, and no more, must be thrown in the remaining three throws (the probability of which is $\frac{n-1 \times 3}{n^3}$ by quest 37); and the probability of throwing it in this order will be $\left(\frac{n-1}{n} \times \frac{n-1 \times 3}{n^3} =\right) \frac{n-1^2 \times 3}{n^4}$.

Therefore the whole probability of throwing two aces, and no more, in four throws will be $\left(\frac{n-1^2 \times 3}{n^4} + \frac{n-1^2 \times 3}{n^4} =\right) \frac{n-1^2 \times 6}{n^4}$; In this case $\left(\frac{25 \times 6}{1296} =\right) \frac{25}{216}$.

QUESTION XXXIX.

The probability of throwing two aces, and no more, in five throws, is required?

SOLUTION.

The probability of throwing an ace the first throw, and throwing but one ace in four throws after, will be

$$\left(\frac{1}{n} \times \frac{n-1^3 \times 4}{n^4} = \right) \frac{n-1^3 \times 4}{n^5}, \text{ by quest. 34.}$$

And the probability of missing an ace the first throw, and throwing it twice, only, in the 4 succeeding throws

$$\text{is } \left(\frac{n-1}{n} \times \frac{n-1^2 \times 6}{n^4} = \right) \frac{n-1^3 \times 6}{n^5} \text{ by quest. 38.}$$

Therefore the whole probability of throwing two aces,

and no more, in five throws will be $\left(\frac{n-1^3 \times 4}{n^5} + \right.$

$$\left. \frac{n-1^3 \times 6}{n^5} = \right) \frac{n-1^3 \times 10}{n^5}; \text{ In this case } \left(\frac{125 \times 10}{7776} = \right)$$

$$\frac{625}{3888}$$

COROL.

Since the probability of throwing two aces, and no more, in

$$\left\{ \begin{array}{l} 3 \\ 4 \\ 5 \end{array} \right\} \text{Throws is } \left\{ \begin{array}{l} \frac{n-1 \times 3}{n^3} \text{ by quest. 37.} \\ \frac{n-1^2 \times 6}{n^4} \quad 38. \\ \frac{n-1^3 \times 10}{n^5} \quad 39. \end{array} \right.$$

Therefore the probability of throwing two aces, and

no more, in m -throws will be $\frac{n-1}{n} \times \frac{n-2}{n} \times \frac{n-3}{n} \times \dots \times \frac{n-m+1}{n}$.

SCHOLIUM. The chance of throwing any two faces at one throw, with two dice, is the same, with that of throwing the same faces, at two operations, with a single die; as will appear from what follows.

All the different throws, upon two dice, are represented in the following table; where the numbers 1. 2. 3. 4. 5. 6. denote the several faces of the one die, and 1. 2. 3. 4. 5. 6. the faces of the other die.

1.1	2.1	3.1	4.1	5.1	6.1
1.2	2.2	3.2	4.2	5.2	6.2
1.3	2.3	3.3	4.3	5.3	6.3
1.4	2.4	3.4	4.4	5.4	6.4
1.5	2.5	3.5	4.5	5.5	6.5
1.6	2.6	3.6	4.6	5.6	6.6

From which table it appears,

1st. That $n=36$ is the whole number of chances on two dice; and consequently that 36 will be the common denominator of all fractions, that express probabilities concerning them.

2^d. That the probability of throwing one ace, or more, with two dice, will be $\frac{11}{36}$; the same as that of throwing one ace, or more, at two throws, with one die, see quest. 30; for, in the table, one or more aces will be found in every one of the six uppermost squares, and in five other squares in the left hand column.

3^d. Also the probability of throwing one ace, and no more, in two throws with one die, which by quest. 32 is $\frac{1}{6}$, will appear to be equal to the probability of throwing only one ace with two dice, which is also $\frac{1}{6}$;
for

for of the 11 chances, for one ace, or more, above-quoted from the table, there is one for two aces.

4th. That the probability of throwing an ace, or a deux, will be $\left(\frac{20}{36} = \right) \frac{5}{9}$; for an ace will be found in every one of the 6 squares of the upper line, a deux will be found in every one of the 6 squares of the second line, an ace will be found in the four remaining squares of the left hand column, and a deux in the 4 remaining squares of the next column thereto.

5th. In like manner the probability of throwing ace, deux, or tre, will be $\left(\frac{27}{36} = \right) \frac{3}{4}$; and that of throwing ace, deux, tre, or quator $\left(\frac{32}{36} = \right) \frac{8}{9}$.

QUESTION XL.

The respective probabilities of throwing any number of points, with two dice, are required.

SOLUTION.

This question may be readily answered, by disposing the numbers of the above table, in the following order.

No of Points.	How they may be thrown.						Chan. thereof.
2	1.1						1
3	1.2	2.1					2
4	1.3	2.2	3.1				3
5	1.4	2.3	3.2	4.1			4
6	1.5	2.4	3.3	4.2	5.1		5
7	1.6	2.5	3.4	4.3	5.2	6.1	6
8	2.6	3.5	4.4	5.3	6.2		5
9	3.6	4.5	5.4	6.3			4
10	4.6	5.5	6.4				3
11	5.6	6.5					2
12	6.6						1
Total of the Chances							36

That is the probability of $\left\{ \begin{smallmatrix} 5 \\ 6 \end{smallmatrix} \right\}$ points, &c. is $\left\{ \begin{smallmatrix} 4 \\ 36 \\ 5 \\ 36 \\ 6 \\ 36 \end{smallmatrix} \right\}$ &c.

SCHOLIUM. It may be worth observing, that if the multinomial, $r+r^2+r^3+r^4+r^5+r^6$, be squared (that is multiplied by itself) the numeral coefficients, annexed to each power of r in the product, will be the number of chances, for the same number of points, as is expressed by the Index of the power, to which they are annexed. See the operation:

$$\begin{array}{r}
 r^2 + 2r^3 + 3r^4 + 4r^5 + 5r^6 + 6r^7 + 5r^8 + 4r^9 + 3r^{10} + 2r^{11} + r^{12} \\
 \hline
 \begin{array}{r}
 r^2 + r^3 + r^4 + r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^3 + r^4 + r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^4 + r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^8 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^9 + r^{10} + r^{11} + r^{12} \\
 + r^{10} + r^{11} + r^{12} \\
 + r^{11} + r^{12} \\
 + r^{12}
 \end{array}
 \end{array}$$

QUESTION XL.

A, being at play at back-gammon, is obliged to make a blot; now his throw is such, that he can make it, either where his adversary, *B*, can take it up with a single ace, or else where he can take it up by throwing seven in any manner: The question is, where he should make the blot?

SOLUTION.

Because the number of chances for throwing one ace or more is 11; and the number of chances for throwing 7, in any manner, are but 6; Therefore, it will be safest to make the blot, where it may be taken up by throwing 7.

QUESTION XLII.

Whether is it safer, to make a blot at back-gammon, where it can be taken up by an ace, or where it can be taken up by a trè?

SOLUTION.

The number of chances for throwing one ace or more, and those for throwing one trè or more, are each eleven; but then, there are two chances for throwing deux ace, or three; therefore it will be safer to make the blot, where it can be taken up, only, by an ace.

The

The following table will shew the chances of taking up a single blot, howsoever situated.

No. of points to hit.	Chances.	Total Chances.
1	11	11
2	11 + 1	12
3	11 + 2	13
4	11 + 3	14
5	11 + 4	15
6	11 + 5	16

No. of points to hit.	Chances.	Total Chances.
7	6	6
8	5	5
9	4	4
10	3	3
11	2	2
12	1	1

C O R O L.

Hence, if the blot is liable to be hit by any one face of the die, the mean probability of hitting it will be

$$\left(\frac{11+16}{2 \times 30} = \frac{27}{72} \right) \frac{3}{8} \text{ nearly.}$$

QUESTION XLIII.

If two blots be made at backgammon, so as to be hit by two different faces of the die, what is the probability of hitting one, or both of them?

SOLUTION.

By the first given table, it will appear that the probability of throwing one, or more, of any two given faces is $\frac{20}{36}$.

But besides that, one or both the blots may be hit by the two dice at length; and the probability of that, is different, according to the number of points that will hit them, as in the following table:

Faces

Faces to hit.	Chances.	Total Chances
1·2	20+1	21
1·3	20+2	22
1·4	20+3	23
1·5	20+4	24
1·6	20+5	25
2·3	20+1+2	23
2·4	20+1+3	24
2·5	20+1+4	25

Faces to hit.	Chances.	Total Chances
2·6	20+1+5	26
3·4	20+2+3	25
3·5	20+2+4	26
3·6	20+2+5	27
4·5	20+3+4	27
4·6	20+3+5	28
5·6	20+4+5	29

C O R O L

Hence the probability of hitting two such blots will, at a medium, be $\left(\frac{21+29}{2 \times 36} = \right) \frac{25}{36}$

Q U E S

QUESTION XLIV.

If there be three blots, so situated as to be hit by three different faces of the die, the probability of hitting one or more of them is required?

SOLUTION.

The first table will give the probability of hitting one, or more, of the blots with a single face, or faces; but beside that, there will be the probability of hitting one, or more, of the blots with two dice, at length; the least of which will be, when the given faces are 1, 2, 3, which have $(2+1=) 3$ such chances; and the greatest, when the given faces are 4, 5, 6, which have $(3+4+5=) 12$ such chances, the medium of which, viz. $\left(\frac{3+12}{2}= \right) \frac{15}{2}$ being added to 27, makes the whole probability, about $\left(27 + \frac{15}{2} = \right) \frac{69}{2}$, which divided by the common denominator 36 becomes $\left(\frac{69}{72} = \right) \frac{23}{24}$.

COROLLARY.

Hence, if a player at backgammon makes 3 blots, which are severally within reach of being hit by a single face of the die, it is almost a certainty that one of them, at least, will be hit.

QUEST

QUESTION XLV.

Suppose two purses, each containing n counters; where a of a are white, and b black: If a person draw a counter out of each purse, what probability hath he to draw one white counter, and no more?

SOLUTION.

The probability of his drawing a black counter out of the first purse, and a white counter out of the second will (by corol. to quest. 29) be $\left(1 - \frac{a}{a+b} \times \frac{a}{a+b}\right)$

$$\frac{a}{a+b} - \frac{aa}{a+b^2} = \frac{a+b \times a - aa}{a+b^2} = \frac{aa+ab-aa}{a+b^2} = \frac{ab}{a+b^2}$$

And the probability of his drawing a white counter out of the first purse, and a black counter out of the second, will, by a like argument, be the same, viz.

$$\frac{ab}{a+b^2}$$

Therefore the probability required will be $\left(\frac{ab}{a+b^2} + \frac{ab}{a+b^2}\right)$

$$\frac{ab}{a+b^2} = \frac{2ab}{a+b^2}$$

COROL.

Hence, if a and b severally represent the number of chances, for the happening and failing of an event, at one

one trial; then shall one, or more, of the terms of the second power of the binomial $a+b^2$ be the numerators of the fractions, which will express the probabilities of all the varieties that can possibly happen, concerning two such events in two trials. And the power itself $a+b^2$ will be the common denominator of the said fractions.

For instance, the probability, in two trials, of the happening of

Two events	} will be {	$\frac{a'a}{a+b^2}$	Quest. 28.
Only one of them		$\frac{2ab}{a+b^2}$	45.
Neither of them		$\frac{bb}{a+b^2}$	Corol. 29.

The sum of which three probabilities $\frac{aa}{a+b^2}$ $\frac{2ab}{a+b^2}$ and $\frac{bb}{a+b^2}$, viz. $\frac{aa+2ab+bb}{aa+2ab+bb}$ is unity, as it evidently ought to be.

Again the probability of one or both of the events happening will be $\left(\frac{aa+2ab}{a+b^2}\right) \cdot \frac{a+b^2-bb}{a+b^2}$; And the probability that both of them will not happen, will be $\left(\frac{2ab+bb}{a+b^2}\right) \cdot \frac{a+b^2-aa}{a+b^2}$.

QUES.

QUESTION XLVI.

If there be three purses, each containing a , white, and b , black counters; and if a person draw one counter out of each purse, what probability is there that they shall be all white?

SOLUTION.

If he draws a white counter out of the first purse (the probability of which is $\frac{a}{a+b}$) then he must draw two white counters out of the two remaining purses (the probability of which is $\frac{aa}{a+b^2}$): but neither of those events will be effectual without the happening of the other, and therefore $\left(\frac{a}{a+b} \times \frac{aa}{a+b^2}\right) = \frac{a^3}{a+b^3}$ will be the probability required.

COROL.

In like manner the probability of drawing three black counters, or of failing in each attempt will be

$$\frac{bbb}{a+b^3}.$$

QUES.

QUESTION XLVII.

If a person draws out of three purses, as in the last question, what is the probability that two of the counters drawn, and no more, shall be white?

SOLUTION.

If he draws a black counter the first time (the probability of which is $\frac{b}{a+b}$) then he must draw two white counters out of the two remaining purses (the probability of which is $\frac{aa}{a+b^2}$), and therefore the probability of

succeeding, by drawing in that manner will be $\left(\frac{b}{a+b} \times \frac{aa}{a+b^2}\right) = \frac{aab}{a+b^3}$

If he draws a white counter the first time (the probability of which is $\frac{a}{a+b}$) then he must draw, only, one white counter out of the two remaining purses (the probability of which is $\frac{2ab}{a+b^2}$ by quest 45). And therefore the probability of succeeding, in this manner, will be $\left(\frac{a}{a+b} \times \frac{2ab}{a+b^2}\right) = \frac{2aab}{a+b^3}$. Therefore the whole

probability of the thing required will be $\left(\frac{aab}{a+b^3} + \frac{2aab}{a+b^3}\right) = \frac{3aab}{a+b^3}$

COROL.

COROL

Hence the probability of drawing two white counters, or more, will be the sum of the results of the two last questions, viz. $\frac{a^3 + 3aab}{a + b^3}$.

QUESTION XLVIII.

If a person draws out of the three purses, as in the two last questions, what is the probability that one white counter, and no more, shall be drawn?

SOLUTION.

If he draws a white counter the first time; then, at the other two trials, he must draw two black ones; the probability of doing both which is $\left(\frac{a}{a+b} \times \frac{bb}{a+b}\right) =$

$$\frac{abb}{a+b^3}$$

If he draws a black counter at the first trial; then, at the other two trials, he must draw one black and one white counter; the probability of doing both which is $\left(\frac{b}{a+b} \times \frac{2ab}{a+b^2}\right) = \frac{2abb}{a+b^3}$.

Therefore the probability required will be $\left(\frac{abb}{a+b^3} + \frac{2abb}{a+b^3}\right) = \frac{3abb}{a+b^3}$.

COROL. I.

The probability of drawing one, or two white counters, neither less than one, nor more than two, will be

$$\left(\frac{3aab}{a+b^3} + \frac{3abb}{a+b^3} \right) \frac{3ab}{a+b^3}.$$

COROL. II.

The probability of drawing one, two, or three white counters, that is, not less than one, will be $\left(\frac{a^3}{a+b^3} + \right.$

$$\left. \frac{3aab}{a+b^3} + \frac{3abb}{a+b^3} \right) \frac{a+b^3-b^3}{a+b^3}.$$

COROL. III.

The probability of drawing none, or, at most, but one, white counter, will be $\frac{3abb+b^3}{a+b^3}.$

COROL. IV.

And the probability of drawing none, one, or at most but two, white counters, will be $\frac{a+b^3-a^3}{a+b^3}.$

COROL. V.

Hence it appears, that one or more of the terms of the binomial $a+b$, raised to the third power, will be the

the numerators of those fractions, which express the probabilities of all the varieties that can possibly happen in three trials; concerning events, the number of chances, for the happening or failing of one of which, are respectively a or b ; and that the common denominator of all those fractions will be $a+b^3$ the power itself.

For instance, the probability, in three trials, of the happening of,

The three events	} will be {	$\frac{a^3}{a+b^3}$	} by {	Quest. 46.
Only two of them		$\frac{3aab}{a+b^3}$		47.
Only one of them		$\frac{3abb}{a+b^3}$		48.
Neither of them		$\frac{b^3}{a+b^3}$		Corol. to 46.

The sum of which four probabilities, viz.
 $\frac{a^3+3aab+3abb+b^3}{a+b^3}$ is unity, as it manifestly should be.

GENERAL COROLLARIES.

1st. Therefore all the questions that can possibly be asked, concerning the happening, or failing, of any number of events, in m trials will (if a expresses the chances for happening, and b the chances for failing) be answered, by the assistance of one, or more, of the terms of the binomial $a+b^m$, as a numerator, and the whole binomial, as a denominator.

That

REPOSITORY.

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3. The probability of the happening of, neither, or but one, or two, or at most of but n , such events will be

$$\frac{b^m + mb^{m-1}a + \frac{m(m-1)}{1 \cdot 2}b^{m-2}a^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}b^{m-3}a^3}{a + b^m}$$

the numerator of which fraction must consist of $n+1$ terms.

And the probability of the happening, at most, but $n-1$ such events will be

$$\frac{a + b^m - a^m}{a + b^m}$$

QUESTION XLIX.

In a lottery, whereof the number of blanks is to the number of prizes as 39 to 1 (such was the lottery of the year 1720) how many tickets must be purchased, that the buyer may have an equal chance for one or more prizes?

SOLUTION.

The probability of having one prize, or more, in m tickets, in a lottery wherein the blanks are to the prizes as 39 to 1, is the same, with that of throwing one ace, or more, in m throws, with a die that has $(1+39=)$ 40 faces: Putting therefore $40=n$, the said probability

will (by corol. quest. 31.) be $\frac{n^m - n^{m-1} - n^{m-2} - \dots - n + 1}{n^m}$,

Or (if $1=a$ and $39=b$) $\frac{a + b^m - b^m}{a + b^m}$. And the
pro:

probability of missing $\frac{b^m}{a+b^m}$. But, by the question, there is to be an equal chance for the having, or missing, one, or more prizes: Now, if the certainty of having or missing one, or more, prizes be denoted by unity, then the probabilities of an equal chance for having, or missing, one, or more, of them, will each of them be denoted by $\frac{1}{2}$.

Therefore
$$\frac{a+b^m-b^m}{a+b^m} = \frac{b^m}{a+b^m} = \frac{1}{2}$$

Whence

$$2b^m = a + b^m.$$

Which in logarithms
will be

$$\left. \begin{array}{l} \text{Log. } 2 + m \times L.b = m \times L.a + b \end{array} \right\}$$

Or

$$\text{Log. } 2 = m \times L.a + b - m \times L.b$$

Th.

$$\frac{\text{Log. } 2}{L.a + b - L.b} = m$$

In this case, $L.a + b = L.40 = 1.60206$

$$L.b = L.39 = 1.59106$$

$$L.a + b - L.b = 0.011000301032736 = m$$

Therefore the number of tickets must be greater than 27.

QUESTION L.

In a pack of 26 cards, 13 of which are black, and 13 red: If m cards be dealt, how many is there an equal chance of being red?

SOLU.

SOLUTION.

If the number of chances for the happening of the event be denoted by a , and those for its failing by b ,

Then b^m ,

$$b^m + m b^{m-1} a,$$

$$b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2,$$

&c.

being severally divided by $b+a$, will express the probability of its not happening
&c. thrice, twice, once.

in m trials.

And because the question requires, how many times the event will happen, in m trials, upon an equality of chance, it will follow, that when the event is to happen

Once, b^m

Twice, $b^m + m b^{m-1} a$

Thrice, $b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2$

&c.

being severally divided by $b+a$ will be equal to $\frac{1}{2}$: See quest. 49.

Or b^m

$$b^m + m b^{m-1} a$$

$$b^m + m b^{m-1} a + \frac{m \cdot m-1}{1 \cdot 2} b^{m-2} a^2$$

&c.

$$= \frac{1}{2} \times \overline{b+a}^m;$$

$$= \frac{1}{2} \times \overline{b+a}^m;$$

$$= \frac{1}{2} \times \overline{b+a}^m;$$

&c.

Whence the number of terms, which are equal to $\frac{1}{2} \times \overline{b+a^m}$ will be the answer to the question.

This being premised, Let $a:b::1:p$;

Then $pa=b$; and $\frac{a}{b} (= \frac{a}{pa} =) \frac{1}{p}$.

Therefore if $1 + \frac{1}{p}$ be substituted for $b+a$ the above expressions will become

$$\begin{aligned}
 1 &= \overline{1 + \frac{1}{p}}^m \times \frac{1}{2} \\
 1 + m \times \frac{1}{p} &= \overline{1 + \frac{1}{p}}^m \times \frac{1}{2} \\
 1 + m \times \frac{1}{p} + \frac{m \cdot m - 1}{1 \cdot 2} \times \frac{1}{p^2} &= \overline{1 + \frac{1}{p}}^m \times \frac{1}{2} \\
 1 + m \times \frac{1}{p} + \frac{m \cdot m - 1}{1 \cdot 2} \times \frac{1}{p^2} + \frac{m \cdot m - 1 \cdot m - 2}{1 \cdot 2 \cdot 3} \times \frac{1}{p^3} &= \overline{1 + \frac{1}{p}}^m \times \frac{1}{2} \\
 &\text{Ec.}
 \end{aligned}$$

Now, in the question before us, where $a=b$, $p=1$; the expression $\overline{1 + \frac{1}{p}}^m$ will become $\overline{1+1}^m$. In which power, the several terms, at either extremity, are equal; and therefore $\overline{1+1}^m \times \frac{1}{2}$ will consist of half the terms in $\overline{1+1}^m$.

But the number of terms in $\overline{1+1}^m = m+1$: Therefore the answer will be $\frac{m+1}{2}$.

COROL.

COROL.

If r represent the number of times that the proposed event is required to happen; then, when there is an equality of chance for its happening or missing, $r = \frac{m+1}{2}$ by the above question. Therefore; m , the number of trials, in which it will be an equal chance whether the event shall happen r times or not, will be $= 2r-1$.

And therefore in a lottery, in which the number of prizes is equal to the number of blanks; if it be required to know how many tickets should be bought, in order to have r , or more, prizes; the answer will be $2r-1$ tickets; that is, in order to have an equal chance to have 1, 2, 3, 4, 5, &c. (or more) prizes, there should be bought 1, 3, 5, 7, 9, &c. tickets.

QUESTION LI.

In a pack of 39 cards, consisting of thirteen hearts, thirteen spades, and thirteen clubs; If m cards be dealt to me, how many may I, on an equality of chance, expect to be hearts?

SOLUTION.

If a , b and p represent the same as in the last question, and r be the number required; then (because there are two chances for a black card to be dealt, and but one

G 2

for

And, when $m =$	1,	1	1	-	-	-	-	-	-	-	= 1, 0:
	2,	1	1	-	-	-	-	-	-	-	= 1, 0:
	4,	2	Terms of the power will be	1	+	$4 \times \frac{1}{2}$	-	-	-	-	= 3, 0:
	5,	2		1	+	$5 \times \frac{1}{2}$	-	-	-	-	= 3, 5:
	7,	3	then	1	+	$7 \times \frac{1}{2}$	+	$\frac{7 \cdot 6}{1 \cdot 2} \times \frac{1}{4}$	-	-	= 9, 75:
	8,	3		1	+	$8 \times \frac{1}{2}$	+	$\frac{8 \cdot 7}{1 \cdot 2} \times \frac{1}{4}$	-	-	= 12, 0:
	10,	4	Terms of the power will be	1	+	$10 \times \frac{1}{2}$	+	$\frac{10 \cdot 9}{1 \cdot 2} \times \frac{1}{4}$	+	$\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \times \frac{1}{8}$	= 32, 25:
	11,	4		1	+	$11 \times \frac{1}{2}$	+	$\frac{11 \cdot 10}{1 \cdot 2} \times \frac{1}{4}$	+	$\frac{11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3} \times \frac{1}{8}$	= 40, 88:

Whence, if $m =$	1,	$\left\{ \begin{array}{c} m \\ 3 \end{array} \right\} \times \frac{1}{2}$	0, 75;	And	1,	$\left\{ \begin{array}{c} m \\ 3 \end{array} \right\} \times \frac{1}{2}$	1, 0.
	2,		1, 125;		1		1, 0.
	4,		2, 53;		2		3, 0.
	5,		3, 79;		2		3, 5.
	7,		8, 54;		3		9, 75.
	8,		12, 81;		3		12, 0.
	10,		28, 83;		4		32, 25.
	11,		43, 24;		4		40, 88.

That is, when $m =$	{	1,	{	then one term is	{	greater	{	than	$\left\{ \begin{array}{c} m \\ 3 \end{array} \right\} \times \frac{1}{2}$
		2,		less		than			
	{	4,	{	then 2 terms are	{	greater	{	than	
		5,		less		than			
	{	7,	{	then 3 terms are	{	greater	{	than	
		8,		less		than			
	{	10,	{	then 4 terms are	{	greater	{	than	
		11,		less		than			
		{	&c.		{	&c.		{	

And (putting $x =$ any number whatsoever)

When m is

$\left\{ \begin{smallmatrix} 3x+1 \\ 3x+2 \end{smallmatrix} \right\}$ then $x+1$ terms are $\left\{ \begin{smallmatrix} \text{greater} \\ \text{lesser} \end{smallmatrix} \right\}$ than $\frac{m}{3} \times \frac{1}{2}$.

But, when the chances for the events not happening r times, in m trials (which are signified by the above series) are lesser than $\frac{1}{2}$ the power; then the chances for the events happening are greater than it; Therefore, when $m=3x+2$; there will be more than an equal chance for the events happening $x+1$ times; therefore, (putting $r=x+1$) we may argue as follows:

Since $m=3x+2$; Th. $\frac{m-2}{3}=x$

And since $r=x+1$; $r-1=x$

Therefore $\frac{m-2}{3}=r-1$,

Or $m-2=3r-3$

Th. $\frac{m+1}{3}=r$.

That is, if m cards are dealt, it is more than an equal chance that there should be $\frac{m+1}{3}$ hearts.

C O R O L.

Since $m-2=3r-3$;

Th. $m=3r-1$.

That is, in a lottery, where there are two blanks to a prize, if it be required to know, how many tickets should be bought, in order to have an equal chance for r prizes, the answer will be $3r-1$ tickets; that is to say, in order to have an equal chance for obtaining 1, 2, 3, 4, 5, &c.

Of prizes, there must be bought 2, 5, 8, 11, 14, &c. tickets.

QUESTION LH.

In a pack of 52 cards; consisting of 13 of each suit; if m cards be dealt to me, how many may I, on an equality of chance, expect to be trumps?

SOLUTION.

The symbols being retained, as before; Then because there be three suits of blanks, to one suit of (trumps or)

prizes; $b=3$, and $a=1$; Therefore $p=3$, and $\frac{p+1}{p}^m \times \frac{1}{2}$

$$= \left[\frac{4}{3} \right]^m = \frac{1}{2}:$$

Also $1 + m \times \frac{1}{3} + \frac{m \cdot m - 1}{1 \cdot 2} \times \frac{1}{9}$, &c. to r terms =

$$\left[\frac{4}{3} \right]^m \times \frac{1}{2}.$$

Now from 0,60206 = Log. of 4,

Take 0,47712 = Log. of 3,

Remains 0,12494 = Log. of $\frac{4}{3}$:

That is, when $m =$ $\left\{ \begin{array}{l} \left\{ \begin{array}{l} 2, \\ 3, \\ 6, \\ 7, \end{array} \right\} \text{ then one term of the } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than} \\ \left\{ \begin{array}{l} 10, \\ 11, \end{array} \right\} \text{ then two terms are } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than} \\ \left\{ \begin{array}{l} 14, \\ 15, \end{array} \right\} \text{ then 3 terms are } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than} \end{array} \right\} \begin{array}{l} \frac{m-1}{2} \\ \times \\ \frac{m}{1+m} \end{array}$

Th. when $m =$

$\left\{ \begin{array}{l} 4x+2 \\ 4x+3 \end{array} \right\}$ then $x+1$ terms $\left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\}$ than $\frac{m}{4} \times \frac{m}{2}$.

Therefore when $m = 4x+3$, there will be something more than an equal chance of the effects happening $x+1$ times.

Then since $m = 4x+3$; $\frac{m-3}{4} = x$:

And since $r = x+1$; $r-1 = x$:

Th. $r-1 = \frac{m-3}{4}$; And $r = \left(\frac{m-3}{4} + 1 \right) = \frac{m+1}{4}$.

That is; if m cards are dealt, it will be more than an equal chance, that there will be $\frac{m+1}{4}$ trumps. Therefore in the game of whist, where 13 cards are dealt, there is more than an equal chance for any particular persons having $\frac{13+1}{4} = \frac{14}{4}$ trumps; And since this is more than an equal chance, if any player has but three trumps, or less; he may justly conclude that his partner has four trumps, or more.

COROL.

Since $r-1 = \frac{m-3}{4},$

Or $4r-4 = m-3;$

Th. $4r-1 = m.$

That is; in a lottery, where there are three blanks to a prize, if it be required to know, how many tickets should be bought, in order to have an equal chance to have r prizes, the answer will be $4r-1$ tickets; thus, in order to have 1, 2, 3, 4, 5, &c. prizes, there must be bought 3, 7, 11, 15, 19, &c. tickets.

QUESTION LIII.

In a lottery, which has four blanks to one prize; if I purchase m tickets, how many prizes may I expect, on an equality of chance?

SOLUTION.

The symbols being retained as before; then, in this case $b=4$ and $a=1$; Therefore $p=4$, and $\left[\frac{p+1}{p}\right]^m \times \frac{1}{p} =$

$$\left[\frac{5}{4}\right]^m \times \frac{1}{4}.$$

$$\text{Also } 1 + m \times \frac{1}{4} + \frac{m \cdot m - 1}{1 \cdot 2} \times \frac{1}{16} \text{ \&c. } (r) = \left[\frac{5}{4}\right]^m \times \frac{1}{4}.$$

Now

REPOSITORY.

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Now from 0,69897 = Log. of 5.

Take 0,60206 = Log. of 4,

Remains 0,09691 the Log. of $\frac{5}{4}$:

And, when $m =$	3,	3 =	0,29073;	1,95,	0,98
	4,	4 =	0,38764;	2,44,	1,22
	8,	8 =	0,77528;	5,96,	2,98
	9,	9 =	0,87219;	7,45,	3,73
	13,	13 =	1,25983;	18,19,	9,10
	14,	14 =	1,35674;	22,74,	11,37
	18,	18 =	1,74438;	55,51,	27,76
	19,	19 =	1,84129;	69,39,	34,70

Also, when $m =$

19,	18,	14,	13,	9,	8,	4,	3,
Then							
4	4	3	3	2	2	1	1
Terms of the power, viz.							
1 + 19x $\frac{1}{4}$ +	1 + 18x $\frac{1}{4}$ +	1 + 14x $\frac{1}{4}$ +	1 + 13x $\frac{1}{4}$ +	1 + 9x $\frac{1}{4}$ +	1 + 8x $\frac{1}{4}$ +	1 + 1x $\frac{1}{4}$ +	1
$\frac{19 \cdot 18}{1 \cdot 2} \times \frac{1}{16}$	$\frac{18 \cdot 17}{1 \cdot 2} \times \frac{1}{16}$	$\frac{14 \cdot 13}{1 \cdot 2} \times \frac{1}{16}$	$\frac{13 \cdot 12}{1 \cdot 2} \times \frac{1}{16}$				
$\frac{19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3} \times \frac{1}{64}$	$\frac{18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3} \times \frac{1}{64}$						
= 31,58:	= 27,81:	= 10,19:	= 9,15:	= 3,25:	= 3,0:	= 1,0:	= 1,0:

That is, when $m =$ $\left\{ \begin{array}{l} \left\{ \begin{array}{l} 3, \\ 4, \end{array} \right\} \text{ Then one term of } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than } \\ \left\{ \begin{array}{l} 8, \\ 9, \end{array} \right\} \text{ then 2 terms are } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than } \\ \left\{ \begin{array}{l} 13, \\ 14, \end{array} \right\} \text{ then 3 terms are } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than } \\ \left\{ \begin{array}{l} 18, \\ 19, \end{array} \right\} \text{ then 4 terms are } \left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\} \text{ than } \end{array} \right\}$

Th. when $m =$

$\left\{ \begin{array}{l} 5x+3 \\ 5x+4 \end{array} \right\}$ then $x+1$ terms of $\left\{ \begin{array}{l} \text{greater} \\ \text{lesser} \end{array} \right\}$ than $\frac{m}{4} \times \frac{1}{2}$
it are

Therefore when $m=5x+4$, there will be something more than an equal chance of the required effects happening $x+1$ times.

Then since $m = 5x+4$; $\frac{m-4}{5} = x$:

And since $r = x+1$; $r-1 = x$:

Th. $r-1 = \frac{m-4}{5}$; And $r = \left(\frac{m-4}{5} + 1 \right) = \frac{m+1}{5}$.

That is, $\frac{m+1}{5}$ prizes may, on an equality of chance, be expected in m tickets.

C O R O L

Since $r-1 = \frac{m-4}{5}$;

Th. $5r-5 = m-4$,

And $5r-1 = m$.

That is, in such a lottery, if it be required to know, how many tickets should be bought, in order to have an equal

equal chance for r prizes, the answer will be $5r-1$ tickets; therefore, in order to have 1, 2, 3, 4, 5, &c. prizes, there should be purchased 4, 9, 14, 19, 24, &c. tickets.

QUESTION LIV.

If a person, playing with a single die, determines to cast it m times; how many times, out of that number, may he, on an equality of chance, undertake to cast an ace: Or, (which is the same thing) if he casts m dice at once, how many of them may he, on an equality of chance, expect to be aces.

SOLUTION.

The symbols being retained as before; then, in this case $b=5$; and $a=1$; Th. $p=5$; and $\frac{p+1}{p} \Big| ^m \times \frac{1}{2} =$

$$\frac{6}{5} \Big| ^m \times \frac{1}{2}.$$

$$\text{Also } 1 + m \times \frac{1}{5} + \frac{m \cdot m - 1}{1 \cdot 2} \times \frac{1}{25} \text{ &c. } (r) = \frac{6}{5} \Big| ^m \times \frac{1}{2}.$$

Now from *Log.* $6 = 0,77815,$

Take *Log.* $5 = 0,69897,$

Remains *Log.* $\frac{6}{5} = 0,07918;$

And,

And, when $m =$	3,	4,	9,	10,	15,	16,	21,	22,
	eq. 111	eq. 111	eq. 111	eq. 111	eq. 111	eq. 111	eq. 111	eq. 111
	$\times 816,000$	$\times 816,000$	$\times 816,000$	$\times 816,000$	$\times 816,000$	$\times 816,000$	$\times 816,000$	$\times 816,000$
	3 =	4 =	9 =	10 =	15 =	16 =	21 =	22 =
	0,23754;	0,31672;	0,71262;	0,79180;	1,18770;	1,26688;	1,66278;	1,74196;
And the number cor- responding, viz.	1,73,	2,07,	5,16,	6,19,	15,41,	18,49,	46,00,	55,20,
	41,	1,2,	1,2,	1,2,	1,2,	1,2,	1,2,	1,2,
	0,87,	1,04,	2,58,	3,10,	7,71,	9,25,	23,00,	27,60,
	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$	$\frac{1}{2} \times \frac{1}{2}$

Also, when $m =$

22,	21,	16,	15,	10,	9,	4,	3,
Then							
4	4	3	3	2	2	1	1
Terms of the power, viz.							
$1 + 22 \times \frac{1}{2} + \frac{22 \cdot 21}{1 \cdot 2} \times \frac{1}{25} + \frac{22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3} \times \frac{1}{125} = 27, 0^{\circ}$	$1 + 21 \times \frac{1}{2} + \frac{21 \cdot 20}{1 \cdot 2} \times \frac{1}{25} + \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} \times \frac{1}{125} = 24, 24^{\circ}$	$1 + 16 \times \frac{1}{2} + \frac{16 \cdot 15}{1 \cdot 2} \times \frac{1}{25} = 9, 0^{\circ}$	$1 + 15 \times \frac{1}{2} + \frac{15 \cdot 14}{1 \cdot 2} \times \frac{1}{25} = 8, 2^{\circ}$	$1 + 10 \times \frac{1}{2} = 3, 0^{\circ}$	$1 + 9 \times \frac{1}{2} = 2, 8^{\circ}$	$1 = 1, 0^{\circ}$	$1 = 1, 0^{\circ}$

That

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That is, if it be required to know, how many tickets should, in such a lottery, be purchased, in order, on an equality of chance, to expect r prizes; the answer will be $6r-2$: Therefore, in order to have 1, 2, 3, 4, 5, &c. prizes, there should be purchased 4, 10, 16, 22, 28, &c. tickets.

QUESTION LV.

In a lottery, wherein the number of blanks is to the number of prizes as b to a ; how many tickets must be purchased to procure an equal chance for p or more prizes?

SOLUTION.

From a careful observation of the corollaries, to the preceeding five questions, it will appear, that the series, which in each corollary expresses the number of tickets, which ought to be purchased, in order to procure an equality of chance for the having 1, 2, 3, 4, 5, &c. prizes, do severally differ by 2, 3, 4, 5, 6; which, in each separate question, is the number of blanks more one; or (since there is supposed only one prize to a certain number of blanks) the number of chances which one ticket hath of being either blank or prize: Thus in corollary to quest.

50	wherein the Lottery was supposed to have	1	1, 3, 5, 7, 9, &c;	2
51	blanks to one prize; the	2	2, 5, 8, 11, 14, &c;	3
52	N ^o . of tickets necessary to the expectation of 1, 2, 3,	3	3, 7, 11, 15, 19, &c;	4
53	4, 5, &c. prizes, was	4	4, 9, 14, 19, 24, &c;	5
54		5	4, 10, 16, 22 28 &c;	6

Therefore we may conclude, that the same thing will happen in all succeeding questions of this sort; and consequently, that if the first term of the series can be obtained, then all the rest may be found by the continual addition of $a+b$.

Now the first term of this series may be obtained by question 49, where the number of tickets, which must be purchased, that the buyer may have an equal chance to have one prize, is $\frac{\text{Log. } 2.}{\text{Log. } a+b - \text{Log. } b}$. Therefore this quotient, if an integer, or the next greater integer, if a fraction, will be the first term of the series: And if we call this quotient q , and put $a+b=s$; then, in order to have an equal chance, for 1, 2, 3, 4, 5, &c. prizes, we must purchase q , $q+s$, $q+2s$, $q+3s$, $q+4s$, &c. tickets; or universally, in order to have an equal chance for p prizes, we must purchase $q+p-1 \times s$ tickets.

QUESTION LVI.

Supposing the decrements of life to be equal; that is, supposing there be n persons alive at any given age, and that one of them will die every year constantly, till they be

be all dead; it is required to find ($N=$) the present value of an annuity of 1 £ . for a life of that age, allowing the purchaser compound interest?

SOLUTION.

Let r be the amount of 1 £ . in one year at compound interest; that is, the sum of 1 £ . and its interest for one year.

Then, because at the end of one, two, three, &c. years, there will be, but $n-1$, $n-2$, $n-3$, &c. persons alive, it will follow from corol. to quest. 26, that the probability of the given life's surviving the first year will be $\frac{n-1}{n}$, that of the given life's surviving the second

year $\frac{n-2}{n}$, that of its surviving the third year $\frac{n-3}{n}$, &c. which expressions may be considered as the values of each payment of the annuity in the parts of 1 £ .

But, because the first, second, third, &c. payments of this annuity are not to be made till the end of the first, second, third, &c. years, it follows, from quest. 144. part 2. vol. 1. that the present value of the first pay-

ment will be

$$\frac{n-1}{nr}$$

That of the second payment

$$\frac{n-2}{nr^2}$$

That of the third payment

$$\frac{n-3}{nr^3} \text{ \&c.}$$

And therefore the value of the whole annuity will be $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} + \frac{n-4}{nr^4} \left(n \right)$; which series may be divided into two other series, viz.

$$\frac{n}{n} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n) = \frac{1}{1} \times \frac{1-p}{r-1} \quad \text{By quest. 15.}$$

$$\frac{1}{n} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} (n) = \frac{1}{n} \times \frac{1-p \times r}{r-1} - \frac{np}{r-1} \quad \text{by 16.}$$

Therefore (N) the value of the required annuity, will be

$$\frac{1-p}{r-1} - \frac{1-p \times r}{n \times r-1} + \left(\frac{np}{n \times r-1} \right) \frac{p}{r-1}$$

$$\text{But } \frac{1-p}{r-1} + \frac{p}{r-1} = \frac{1}{r-1}$$

$$\text{Therefore } N = \frac{1}{r-1} - \frac{1-p \times r}{n \times r-1} + \frac{np}{n \times r-1}$$

$$\text{Or } N = \frac{n \times r-1}{n \times r-1} - \frac{1-p \times r}{n \times r-1} + \frac{np}{n \times r-1}$$

$$\text{That is } N = \frac{n \times r-1 + np - 1 + p \times r}{n \times r-1}$$

$$\text{Th. } N = \frac{n \times r-1 + p \times r - n}{n \times r-1}$$

where p is the present worth of 1 £ due at the end of n years.

This expression produces the easiest numerical process, but that, below derived, will be hereafter wanted.

$$\text{If } \frac{1}{r-1} = P, \text{ and } \frac{r}{r-1} = Q;$$

$$\text{Then } \frac{1}{r-1} - \frac{1-p \times r}{n \times r-1} \text{ will become}$$

$$P - \frac{1-p}{n} \cdot Q;$$

That

$$\text{That is } P = 2 \times \frac{1}{n} - \frac{p}{n},$$

$$\text{Or } P + 2 \times \frac{1}{n} + \frac{1}{n^2}.$$

EXAMPLE I.

If, at the age of fifty-four years, there are forty-three persons alive; and that, from that time, the decrements of life are equal; what is the value of an annuity of 1*l.* for a life of that age, allowing 4 *per Cent.* compound interest?

$$\text{Here } n = 43 \quad p = 0,185168; \text{ And } r = 1,04$$

$$n = 43 \quad n - 1 + p = 42,185168$$

$$r = 1,04 \quad r = 1,04$$

$$1,72 \quad 43,872574$$

$$,04 \quad -n = 43$$

$$,0688 = n \times r - 1^4,0688) \quad 0,872574(12,683 = N.$$

EXAMPLE II.

If, at the age of fifty-four years, there are thirty-two persons alive, (the decrements of life being supposed equal) what is the value of an annuity of 1*l.* on that life, allowing 4 *per Cent.* compound interest?

$$\text{Here } n = 32 \quad p = 0,285058; \text{ And } r = 1,04$$

$$n = 32 \quad n - 1 + p = 31,285058$$

$$r = 1,04 \quad r = 1,04$$

$$1,28 \quad 32,536460$$

$$,04 \quad -n = 32,0$$

$$,0512 = n \times r - 1^2,0512) \quad 0,536460(10,478 = N.$$

E X.

EXAMPLE III.

Supposing that, at the age of 66 years, there are 20 persons alive, what will the value of an annuity of 1 *l.* on that life, be worth, allowing interest, @ *c.* as above?

Here $n=20$; $p=0,456387$; And $r=1,04$

$n=20$	$n-1+p=19,456387$
$r-1=,04$	$1,04$
<hr/>	<hr/>
,80	20,234642
,04	- n 20,0
<hr/>	<hr/>
,0320	,0320) 0,234642 (7,333= N .

SCHOLIUM. The reader will observe, that, in these three examples, the number of persons supposed to be alive at the several ages of 43, 54 and 66, are 43, 32 and 20; Whence it will appear, that in either case (if the decrements of life be equal) all the lives will be extinct at 86 (for $43+43$; $54+32$; and $66+20=86$) which age, the justly celebrated Mr. *de Moivre* has assumed as the utmost probable extent of life. His words are these:

“ Another thing was necessary to my calculation,
 “ which was, to suppose the extent of life confined to a
 “ certain period of time, which I supposed to be at 86:
 “ What induced me to assume that supposition was, 1st.
 “ That Dr. *Halley* terminates his table of observations at
 “ the 84th year; for although out of 1000 children of
 “ one year of age, there are twenty, who, according to
 “ Dr. *Halley*’s tables, attain to the age of 84 years, yet
 “ that number is inconsiderable, and would still have
 “ been reduced, if the observations had been carried two
 “ years farther. 2^d. It appears from the tables of
 “ *Graunt*,

" *Graunt*, who printed the first edition of his book about
 " 80 years ago, that out of 100 new born children, there
 " remained not one after 86 years; this was deduced
 " from the observations of several years, both in the
 " city, and in the country, at the time when, the city
 " being less populous, there was a greater facility of
 " coming at the truth, than at present. 3d. I was far-
 " ther confirmed in my hypothesis, by tables of obser-
 " vations made, in *Switzerland*, about the beginning of
 " this century, wherein the limit of life is placed at 86:
 " As for what is alledged, that by some observations, of
 " late years, it appears that life is carried to 90, 95,
 " and to 100 years; I am no more moved by it, than
 " by the examples of *Parr* or *Jenkins*, the first of which
 " lived 152 years, and the other 167."

In conformity, also, to that excellent gentleman, the number of persons supposed to be alive at any age, being (as above observed) the differences between the given age and 86, shall be hereafter called the complement of life. But if the reader shall, for any good reason, think that 86 ought not to be taken for the outmost extent of life, and can pitch on a truer number, then, in all the subsequent rules, such number must be substituted in the room of 86.

As the above rule for finding the value of a single life, may frequently be wanted in practice, it is thought expedient to annex it, in words at length, as follows:

The rule to find the present value of an annuity of 1 l. which is to continue during the life of a person of a given age, allowing compound interest at a given rate, and supposing the decrements of life to be equal:

Let the number of years, which the person wants of eighty-six, be called the complement of life; and let the sum of one pound, and its Interest for one year, be called the rate.

Seek

Seek, in the tables, for the present worth of one pound, due at the end of the complement of life; to which add the complement of life less one.

Multiply the above sum by the rate, and from the product take the complement of life; reserving the remainder for a dividend

Multiply the interest of one pound for one year by itself, and that product by the complement of life, for a divisor.

Then, the quotient of that division, will be the value of the annuity required.

E X A M P L E.

What is the value of an annuity of 1*l*. for the life of a person of ten years of age, allowing compound interest at 4 per Cent.

Here $(86 - 10 =) 76$ is the complement of life.

And $(1 \div 100 =) 1,04$ is the rate.

The present worth of 1*l*. due at the end of } 0,05075
76 years

The complement of life less one is 75,00000

Their sum is 75,05075

Which being multiplied by the rate 1,04

Will produce 78,05278

From which subtracting the complement } 76,00000
of life

There will remain for a dividend 2,05278

If the interest of 1*l*. viz. 0,04 be multiplied by itself, the product will be 0,0016. which multiplied by 76 will produce 0,1216 for a divisor.

And if 2,05278 be divided by 0,1216 the quotient will be 16,8814 the value of the annuity required.

N. B.

N. B. The value of an annuity of 1*l.* being multiplied by the yearly income of any other annuity, for the same life or lives, will give the value thereof: That is, the answers found to this, and the subsequent questions, may be considered as the number of years purchase, which such annuities, as are described in the several questions, are worth.

This computation will be rendered somewhat easier, by using the table in page 77. Thus,

Take the present worth of one pound, due at the end of the complement of life, from unity; multiply the remainder by the number, which (in the above mentioned table) stands on a line, with the rate of interest, under the letter Q; and divide this product by the complement of life.

Subtract the above-found quotient from the number, which (on the same line of the same table) stands under the letter P; and the remainder will be the value required.

Thus in the above example,

The tabular number under *Q* is 650,

That under *P* is 25;

And if $(1 - 0,05075 =) 0,94925$ be multiplied by 650, the product will be 617,013; which being divided by 76, will quote 8, 1186.

And if 8, 1186 be subtracted from 25, the remainder (*viz.* 16,8814) will be the value of the annuity, as before.

QUES.

QUESTION LVII.

The decrements of life being equal, suppose A, the complement of whose life is m , is to pay to B, or his assigns, an annuity of 1*l. per Annum*, for n years certain, if A should live so long; what is the present value of B's annuity?

Since every payment of B's annuity, depends upon the continuance of A's life, it will follow (by reasoning as in quest. 56) that it will be worth n terms of the series $\frac{m-1}{mr} + \frac{m-2}{mr^2} + \frac{m-3}{mr^3} + \frac{m-4}{mr^4}$, &c. which may be divided into the two following series, *viz.*

$$\frac{m}{m} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n) = 1 - \frac{1}{r^n} \times \frac{1}{r-1}; \text{ And}$$

$$- \frac{1}{m} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n) = 1 - \frac{1}{r^n} \times \frac{r}{m \times r-1} - \frac{1}{r^n} \times \frac{n}{m \times r-1}$$

Therefore the value of the required annuity will be,

$$1 - \frac{1}{r^n} \times \frac{1}{r-1} + \frac{n}{m r^n} \times \frac{1}{r-1} = 1 - \frac{1}{r^n} \times \frac{r}{m \times r-1}:$$

Now $\frac{1}{1} - \frac{n}{m} = \frac{m-n}{m}$; Th. it will be

$$\text{come } 1 - \frac{m-n}{m r^n} \times \frac{1}{r-1} = 1 - \frac{1}{r^n} \times \frac{r}{m \times r-1}.$$

EXAMPLE.

Suppose A, whose age is 43, is to pay to B an annuity of 1*l.* for 32 years certain, if he (that is A) should live

so long: What is the present worth of B's annuity, allowing 4 per Cent. compound interest?

$$\text{Here } m = (86 - 43) = 43; n = 32; \frac{1}{r^n} = 0,285058; \&c = 1,04$$

$$\begin{array}{r} m = 43 \quad m = 43; \quad m - n = \frac{11}{m = 43} \\ r - 1 = 0,04 \quad n = 32 \quad 3,135638 \\ \hline 1,72m - n = 11 \\ \hline 0,04 \\ \hline 0,0688 \end{array}$$

$$\begin{array}{r} 1,0 \\ \hline 1 - 0,072922 \\ \hline 1 - \frac{m-n}{m^n} = 0,927078 \end{array} \quad \begin{array}{r} 1 \\ \hline 1 - 0,285058 \\ \hline 1 - \frac{1}{r^n} = 0,714942 \end{array}$$

$$\begin{array}{r} 25 = \frac{1}{r-1} \quad 1,04 \\ \hline 23,17695 \quad c,688 \\ \hline 10,807 \quad 0,743540 \\ \hline 12,370 \end{array} \quad \begin{array}{r} 10,807 \\ \hline 10,807 \end{array}$$

The Answer.

QUESTION LVIII.

It is required to find the value of an annuity of 1 £. to continue s years, if a person of a given age shall live so long; allowing compound interest at a given rate, and supposing that (by a table of observations deduced from the bills of mortality of the place where the annuitant resides) it should appear, that the number of persons living at the beginning, and end, of that period of time, (during which the annuity is to continue) are proportional to the numbers a and b , and that the decrements of life, are equal?

SOLU-

SOLUTION.

Since the decrements of life are equal, during the continuance of the proposed annuity, 'tis evident that a and b must be the extreme terms of $s+1$ numbers in arithmetical progression; and therefore, the common difference of the numbers in that progression will be $\frac{a-b}{s}$ by quest. 6.

part 2. vol. I.) That is, $a - \frac{a-b}{s}$, $a - \frac{2a-2b}{s}$, $a - \frac{3a-3b}{s}$, &c.

Or $\frac{sa-a+b}{s}$, $\frac{sa-2a+2b}{s}$, $\frac{sa-3a+3b}{s}$, &c.

will represent the numbers themselves; and, if we argue as in quest. 56, it will appear that the probabilities of the continuance of the given life for 1, 2, 3, &c. years, will be $\frac{sa-a+b}{sa}$, $\frac{sa-2a+2b}{sa}$, $\frac{sa-3a+3b}{sa}$, &c. the present worths of which probabilities, being added together, give $\frac{sa-a+b}{sar} + \frac{sa-2a+2b}{sar^2} + \frac{sa-3a+3b}{sar^3}$ (&c.) for the value of the annuity required.

Now the above may be divided into the two following series, viz.

$$\begin{aligned} \frac{sa}{sa} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (s) &= \frac{1}{sa} \times \frac{sa}{r-1} - \frac{sa}{r-1 \times r^3} \\ - \frac{a-b}{sa} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} (s) &= \\ \frac{1}{sa} \times \left\{ \begin{aligned} &-\frac{r \times a - b}{r-1} + \frac{s \times a - b}{r-1 \times r^3} \\ &+ \frac{r \times a - b}{r-1^2 \times r^3} \end{aligned} \right\} \end{aligned}$$

And therefore, the value of the annuity will be

$$\frac{1}{sa} \times \frac{sa}{r-1} - \frac{r \times a - b}{r-1} - \frac{sb}{r-1 \times r^s} + \frac{r \times a - b}{r-1 \times r^s}$$

QUESTION LIX.

It is required to find the value of an annuity to continue $s+t$ years, if a person of a given age shall live so long; allowing compound interest at a given rate, and supposing that (by a table of observations deduced from the bills of mortality of the place where the annuitant resides) it should appear that the number of persons living, at the beginning, at the end of s years, and at the end of $s+t$ years, are respectively proportional to the numbers a , b , and c ; and that the decrements of life, for the separate intervals of s and t years, are equal; and lastly, that a is the difference between the common differences, of the two arithmetical progressions, of which a and b , and b and c , are the extreme terms. That is

$$a = \frac{a-b}{s} - \frac{b-c}{t}?$$

SOLUTION.

The value of the annuity for the first s years will be

$$\frac{1}{sa} \times \frac{sa}{r-1} - \frac{r \times a - b}{r-1} - \frac{sb}{r-1 \times r^s} + \frac{r \times a - b}{r-1 \times r^s}, \text{ Or}$$

$$\frac{1}{r-1} - \frac{r}{r-1} \times \frac{a-b}{sa} - \frac{1}{r-1 \times r^s} \times \frac{b}{a} + \frac{r}{r-1} \times \frac{a-b}{sa}$$

by quest. 58. and the probabilities of the given life's continuing

continuing $s+1$, $s+2$, $s+3$, &c. years will (by arguing as before) be $\frac{tb-b+c}{ta}$, $\frac{tb-2b+2c}{ta}$, $\frac{tb-3b+3c}{ta}$, &c. whence the value of the annuity for the last t years will be represented by the series $\frac{tb-b+c}{tar^{s+1}} + \frac{tb-2b+2c}{tar^{s+2}} + \frac{tb-3b+3c}{tar^{s+3}} (t)$, which being summed in the same manner as the last will amount to

$$\frac{1}{tar^s} \times \frac{tb}{r-1} - \frac{r \times b - c}{r-1} - \frac{tc}{r-1 \times r^t} + \frac{r \times b - c}{r-1 \times r^t}, \text{ Or}$$

$$\frac{1}{r-1 \times r^s} \times \frac{b}{a} - \frac{r}{r-1^2 \times r^s a} \times \frac{b-c}{t} - \frac{c}{r-1 \times ar^{s+t}} + \frac{r \times b - c}{r-1^2 \times tar^{s+t}}$$

The sum of which two values will be the value of the whole annuity: Now it is observable, that the third term of the value of the annuity for the first s years is the same with the first term of the value for the last t years, but with a contrary sign, therefore those terms will vanish out of the sum; again the fourth term of the former hath the same common factor with the second term of the latter (*viz.* $\frac{r}{r-1^2 \times r^s a}$), those terms have contrary signs, and the quantities multiplied into that factor, are $\frac{a-b}{s}$ and $\frac{b-c}{t}$, which have a given difference, *viz.* a , therefore those two terms may be expressed by $\frac{ra}{r-1^2 \times r^s a}$; And the value of the whole annuity will be

$$\frac{1}{r-1} - \frac{r}{r-1} \times \frac{a-b}{sa} + \frac{ra}{r-1} \times \frac{1}{r^s a} - \frac{c}{r-1} \times \frac{1}{rar^s + s} \\ \left(\frac{r \times b-c}{r-1} \times \frac{1}{tar^s + s} \right)$$

QUESTION LX.

Let it be required to find the value of an annuity to continue $s+t+v$ years, if a person of a given age shall live so long; allowing compound interest at a given rate, and supposing that (by a table of observations deduced from the bills of mortality of the place where the annuitant resides) it should appear that the number of persons living at the beginning, at the end of s years, at the end of $s+t$ years, and at the end of $s+t+v$ years, are respectively proportional to a , b , c , and d ; that the decrements of life are severally equal during the respective intervals of s , t , and v years, and lastly that

$$\frac{a-b}{s} = \frac{b-c}{t} \text{ and } \frac{b-c}{t} = \frac{c-d}{v} = \&c$$

SOLUTION.

The value of the annuity for the first $s+t$ years being found in the last question, it remains only to find its value for the last v years: Now the probabilities of the given life's continuing $s+t+1$, $s+t+2$, $s+t+3$, &c. years are severally

$$\frac{a-c+d}{va}, \frac{a-2c+2d}{va}, \frac{a-3c+3d}{va}, \&c. \text{ and therefore}$$

the value of the annuity, for the last interval of v years will

will be $\frac{vc-t+d}{var^s+t+1} + \frac{vc-2c+2d}{var^s+t+2} + \frac{vc-3c+3d}{var^s+t+3} \cdot (v)$

which, being summed in the same manner as the former, will become,

$$\frac{1}{var^s+t} \times \frac{vc}{r-1} - \frac{rxc-d}{r-1^2} - \frac{vd}{r-1 \times r^v} + \frac{rxc-d}{r-1^2 \times r^v}, \text{ Or}$$

$$\frac{1}{r-1 \times r^s+t} \times \frac{c}{a} - \frac{r}{r-1^2 \times ar^s+t} \times \frac{c-d}{v} - \frac{d}{r-1 \times ar^s+t+v}$$

$$\left(+ \frac{rxc-d}{r-1^2 \times var^s+t+v} \right)$$

And if we proceed to add the two first terms of this expression, and the two last terms of the result of the last question together, in the same manner as before, the sum will be $\frac{r\beta}{r-1^2 \times ar^s+t}$; and consequently the value of the whole annuity will be

$$\frac{1}{r-1} - \frac{r}{r-1^2} \times \frac{a-b}{sa} + \frac{ra}{r-1^2 \times ar^s} + \frac{r\beta}{r-1^2 \times ar^s+t}$$

$$\left(\frac{d}{r-1 \times ar^s+t+v} + \frac{rxc-d}{r-1^2 \times var^s+t+v} \right)$$

COROL. I.

The manner of continuing this process is so evident, that it seems quite needless to pursue it farther, we shall therefore proceed to a general expression thereof.

Now the above will, by reduction, become

$$\frac{1}{r-1} \times \frac{r-1 \times a}{1} - \frac{a-b \times r}{s} + \frac{a}{rs} + \frac{\beta}{r^s+t} - \frac{r-1 \times d}{r^s+t+v} + \left(\frac{c-d \times r}{wr^s+t+v} \right)$$

Or (putting $P = \frac{1}{r-1}$)

$$PP \times \frac{a}{P} - \frac{a-b \times r}{s} + \frac{a}{rs} + \frac{\beta}{r^s+t} + \frac{d}{P r^s+t+v} + \frac{c-d \times r}{w r^s+t+v}$$

If therefore from any table of observations (deduced from bills of mortality) it appears, that the numbers which are proportional to the living at the end of each year, do for s years decrease in an arithmetical progression; for the next t years, in another arithmetical progression; for the next v years in a third; and for the next w years in a fourth; and so on to the last x years wanted; and if the said tabular numbers, at the beginning of each such period be, severally $a, b, c, \&c.$ to g ; b , being the tabular number at the end of the period s ; then will the common differences of those arithmetical progressions be

$\frac{a-b}{s}, \frac{b-c}{t}, \frac{c-d}{v}, \frac{d-e}{w}, \&c.$ to $\frac{g-b}{x}$: And if, for the

several differences of those differences, we write $\alpha, \beta, \gamma,$

$\&c.$ to x ; viz. $\frac{a-b}{s} - \frac{b-c}{t} = \alpha; \frac{b-c}{t} - \frac{c-d}{v} = \beta$

$\&c.$ Then, an annuity to continue $s+t+v, \&c. + x$ years, (the number of which intervals is w) if a life of that age, (which in the table of observations corresponds to the number of living a) shall continue so long, will be worth

PP

$$\frac{PP}{a} \times \left\{ \frac{\frac{a}{P} - \frac{a-b \times r}{s}}{P, s + t \&c. + z} + \frac{b}{P, s + t \&c. + z} + \frac{\frac{g-b \times r}{z}}{P, s + t \&c. + z} \right. \\ \left. + \frac{a}{r, s} + \frac{\beta}{r, s + t} + \frac{\gamma}{r, s + t + v} + \frac{\delta}{r, s + t + v + w} (m-1) \right\}$$

Where $a, \beta, \gamma, \&c.$ will be found, most commonly, to denote $+1$ or -1 , and their factors, $\frac{1}{r, s}, \frac{1}{r, s + t}$

$\&c.$ are the numbers, which in a table of the present worths of $1/$ at the given rate, stand against the numbers, $s, s + t, \&c.$

QUESTION LXI.

It is required to find the value of an annuity to continue during a life of a given age, supposing, as before, that by the table of observations, deduced from the Bills of mortality of the place where the annuitant resides, it should appear, that the numbers, proportional to the persons living of the succeeding ages may be divided into several such arithmetical progressions as are above described, and allowing compound interest at a given rate?

SOLUTION.

If the same symbols be retained, as in the last question; Then, since the value of an annuity for the whole life of that age is required, b will necessarily be incon-

H 5 siderable,

fiderable, and consequently, $\frac{g-b}{x}$ will equal $\frac{g}{x}$, which cannot greatly differ from unity; whence such an annuity may be expressed by

$$\frac{PP}{a} \times \frac{a}{P} - \frac{a-b \times r}{s} + \frac{a}{r^s} + \frac{\beta}{r^s+t} + \frac{\gamma}{r^s+t+u} (m).$$

But $\frac{a-b}{s}$ is the common difference of the first arithmetical progression, that is, the difference between the two first tabular numbers, a and the next lesser, which difference may be seen by inspection on the table, and may be denoted by D : and then the annuity will be worth

$$\frac{PP}{a} \times \frac{a}{P} - Dr + \frac{a}{r^s} + \frac{\beta}{r^s+t} + \frac{\gamma}{r^s+t+u} (m).$$

C O R O L L.

Hence if (by any table of such observations) it should appear, that, in the numbers proportional to the living, the several arithmetical progressions should each consist of an equal number of terms; and that their common differences should, also, be in arithmetical progression; then the annuity would be expressed by

$$\frac{PP}{a} \times \frac{a}{P} - Dr + \frac{a}{r^s} + \frac{a}{r^{2s}} + \frac{a}{r^{3s}} (m).$$

But $\frac{a}{r^s} + \frac{a}{r^{2s}} + \frac{a}{r^{3s}}$, &c. is a geometrical progression,

gression, whose greatest term is $\frac{a}{r^s}$, ratio r^s , and number of terms m ; therefore the sum thereof will (by quest.

95. part 2. vol. I.) be $\frac{r^{sm} - 1 \times a}{r^s \times r^s - 1 \times r^{sm} - 1}$, and consequently the annuity will be worth

$$\frac{PP}{a} \times \frac{a}{p} - Dr + \frac{r^{sm} - 1 \times a}{r^s \times r^s - 1 \times r^{sm} - 1}.$$

COROL. II.

If the numbers, proportional to the living, should be a series whose second differences are equal; then will the value of the annuity be represented by

$$\frac{PP}{a} \times \frac{a}{p} - Dr + \frac{a}{r} + \frac{a}{r^2} + \frac{a}{r^3} (m).$$

Where m would represent the same thing as was (in quest. 56) called the complement of life, and the annuity will (by quest. 15) become

$$\frac{PP}{a} \times \frac{a}{p} - Dr + \frac{a}{r-1} - \frac{a}{r-1 \times r^m}.$$

But as the two latter cases will probably seldom occur, it may be sufficient, here, to give a rule in words at length for the former, and more general case; in order to which, it may be convenient to give some directions, concerning the disposition of a table of observations.

Let then the table of observations consist of four columns.

In that column, which is next to the left hand, insert the numbers 1, 2, 3, &c. to represent the years of life, from infancy to the extremity of old age.

In the second column, insert those numbers which (from the bills of mortality of the place) appear to be proportional to the number of persons living, of each particular age, contained in the first column; the first number, therefore, of this column will be the greatest, and every preceding number will be greater than the succeeding; untill the last, which will be the least.

In the third column, let the differences, between the numbers of the second column, be placed, in such a manner, that the difference between any number, and the next lesser may stand on a line with the greater number; those numbers will be proportional to the number of persons, which die between the age, against which they stand, and the next following age, mentioned in the table: When the numbers, which compose this column, continue the same in one, two, three, four, &c. successive ages, they constitute such an interval of years, as has been above represented by the symbols s, t, v, &c. Let therefore every such interval be (by something remarkable, namely, a rule, or larger space, than common) distinguished from the preceding.

In the fourth column, let there be placed, on the first line of each interval, the difference between the last number in the third column of the former interval, and the first number in the same column of the present interval; placing before it the sign $+$, if the number in the former interval exceeds that in the present; and the sign $-$, if the number in the former interval be less than that in the present:

sent: These numbers, with their signs, are above denoted by α , β , γ , &c.

The following TABLE of OBSERVATIONS, deduced (by the very ingenious Mr. Tho. Simpson) from the Bills of Mortality of LONDON, being put into the above-prescribed Form, may serve as an Example.

Ages born	Persons living 1280	D	α
		410	
1	870	170	+ 240
2	700	65	+ 105
3	635	35	+ 30
4	600	20	+ 15
5	580	16	+ 4
6	564	13	+ 3
7	551	10	+ 3
8	541	9	+ 1
9	532	8	+ 1
10	524	7	+ 1
11	517	7	
12	510	6	+ 1
13	504	6	
14	498	6	
15	492	6	
16	486	6	
17	480	6	
18	474	6	
19	468	6	
20	462	7	- 1
21	455	7	
22	448	7	
23	441	7	

Ages born	Persons living	D	α
24	434	8	- 1
25	426	8	
26	418	8	
27	410	8	
28	402	8	
29	394	9	- 1
30	385	9	
31	376	9	
32	367	9	
33	358	9	
34	349	9	
35	340	9	
36	331	9	
37	322	9	
38	313	9	
39	304	10	- 1
40	294	10	
41	284	10	
42	274	10	
43	264	9	+ 1
44	255	9	
45	246	9	
46	237	9	
47	228	8	+ 1
48	220	8	
49	212	8	

Ages

Let then the table of observations consist of four columns:

In that column, which is next to the left hand, insert the numbers 1, 2, 3, &c. to represent the years of life, from infancy to the extremity of old age.

In the second column, insert those numbers which (from the bills of mortality of the place) appear to be proportional to the number of persons living, of each particular age, contained in the first column; the first number, therefore, of this column will be the greatest, and every preceding number will be greater than the succeeding; untill the last, which will be the least.

In the third column, let the differences, between the numbers of the second column, be placed, in such a manner, that the difference between any number, and the next lesser may stand on a line with the greater number; those numbers will be proportional to the number of persons, which die between the age, against which they stand, and the next following age, mentioned in the table: When the numbers, which compose this column, continue the same in one, two, three, four, &c. successive ages, they constitute such an interval of years, as has been above represented by the symbols s, t, v, &c. Let therefore every such interval be (by something remarkable, namely, a rule, or larger space, than common) distinguished from the preceding.

In the fourth column, let there be placed, on the first line of each interval, the difference between the last number in the third column of the former interval, and the first number in the same column of the present interval; placing before it the sign $+$, if the number in the former interval exceeds that in the present; and the sign $-$, if the number in the former interval be less than that in the present:

sent: These numbers, with their signs, are above denoted by α , β , γ , &c.

The following TABLE of OBSERVATIONS, deduced (by the very ingenious Mr. Tho. Simpson) from the Bills of Mortality of LONDON, being put into the above-prescribed Form, may serve as an Example.

Ages born	Persons living 1280	D	α	Ages born	Persons living	D	α
		410		24	434	8	— 1
1	870	170	+ 240	25	426	8	
2	700	65	+ 105	26	418	8	
3	635	35	+ 30	27	410	8	
4	600	20	+ 15	28	402	8	
5	580	16	+ 4	29	394	9	— 1
6	564	13	+ 3	30	385	9	
7	551	10	+ 3	31	376	9	
8	541	9	+ 1	32	367	9	
9	532	8	+ 1	33	358	9	
10	524	7	+ 1	34	349	9	
11	517	7		35	340	9	
12	510	6	+ 1	36	331	9	
13	504	6		37	322	9	
14	498	6		38	313	9	
15	492	6		39	304	10	— 1
16	486	6		40	294	10	
17	480	6		41	284	10	
18	474	6		42	274	10	
19	468	6		43	264	9	+ 1
20	462	7	— 1	44	255	9	
21	455	7		45	246	9	
22	448	7		46	237	9	
23	441	7		47	228	8	+ 1
				48	220	8	
				49	212	8	

Ages born	Persons living	D	α	Ages born	Persons living	D	α
50	204	8		73	54	5	
51	196	8		74	49	4	+
52	188	8		75	45	4	
53	180	8		76	41	3	+
54	172	7	+				
55	165	7		77	38	3	
56	158	7		78	35	3	
57	151	7		79	32	3	
58	144	7		80	29	3	
59	137	7		81	26	3	
60	130	7		82	23	3	
61	123	6	+				
62	117	6		83	20	3	
63	111	6		84	17	3	
64	105	6		85	14	2	+
65	99	6		86	12	2	
66	93	6		87	10	2	
67	87	6		88	8	2	
68	81	6		89	6	1	+
69	75	6		90	5	1	
70	69	5	+				
71	64	5		91	4	1	
72	59	5		92	3	1	
				93	2	1	
				94	1	1	

The author, mentioned in the title of this table, has carried his numbers no farther than 80 years; but another noted author, who very nearly agrees with the former in every age above 25, has continued his numbers lower in the above manner.

The following TABLE of OBSERVATIONS, deduced (by the justly celebrated Dr. HALLER) from the Bills of Mortality of BRESLAW, is as a farther Example, inserted in the Form above recommended.

Ages	Persons living	D	α	Ages	Persons living	D	α
1	1000	145		27	533	7	
2	855	57	+ 88	28	546	7	
3	798	38	+ 19	29	539	8	1
4	760	28	+ 10	30	531	8	
5	732	22	+ 6	31	523	8	
6	710	18	+ 4	32	515	8	
7	692	12	+ 6	33	507	8	
8	680	10	+ 2	34	499	9	1
9	670	9	+ 1	35	490	9	
10	661	8	+ 1	36	481	9	
11	653	7	+ 1	37	472	9	
12	645	6	+ 1	38	463	9	
13	640	6		39	454	9	
14	634	6		40	445	9	
15	628	6		41	436	9	
16	622	6		42	427	10	1
17	616	6		43	417	10	
18	610	6		44	407	10	
19	604	6		45	397	10	
20	598	6		46	387	10	
21	592	6		47	377	10	
22	586	7	- 1	48	367	10	
23	579	6	+ 1	49	357	11	1
24	573	6		50	346	11	
25	567	7	- 1	51	335	11	
26	560	7		52	324	11	
				53	313	11	
				54	302	10	+ 1

Age

Ages	Persons living	D	α
55	292	10	
56	282	10	
57	272	10	
58	262	10	
59	252	10	
60	242	10	
61	232	10	
62	222	10	
63	212	10	
64	202	10	
65	192	10	
66	182	10	
67	172	10	
68	162	10	
69	152	10	
70	142	11	— 1
71	131	11	
72	120	11	
73	109	11	

Ages	Persons living	D	α
74	98	10	+
75	88	10	
76	78	10	
77	68	10	
78	58	9	+
79	49	8	+
80	41	7	+
81	34	6	+
82	28	5	+
83	23	4	+
84	19	4	
85	15	4	
86	11	3	+
87	8	3	
88	5	2	+
89	3	2	
90	1	1	+

The numbers, proportional to the persons living, in this table were copied from Dr. Halley's table, as inserted at the end of Mr. De Moivre's doctrine of chances, which is carried no farther than 84 years; against which the number 20 is placed, where I have here inserted 19; which alteration I made, because, otherwise the number in the fourth column, even with 83, would have been 2, contrary to the general law observed, in all other instances, both in this and the *London* table, after the age of 8 years: The numbers, placed against the following years, were inserted in that manner, which I conceived to be most conformable to that general law; but if I have erred therein, the value of a life will be but very little affected thereby.

The

The RULE,

To find the present value of an annuity of 1 l. to continue during a single life of a given age, allowing compound interest at a given rate, by the assistance of a table of observations (deduced from the bills of mortality of the place where the annuitant resides) disposed in the manner above described, and a table of the present worths of 1 l. sterling, due at the end of any number of years to come.

From the first number of the left hand column of each interval, which follows the given age in the tables of observations, take the given age, and let the remainders be called the complements of each interval; also let 1 l. and its interest be called the rate.

As a title of distinction, between the two hereafter directed columns of figures, place the signs + and —.

Multiply severally the present worths of 1 l. due at the ends of those numbers of years, which are expressed by the respective complements of each of the above-mentioned intervals, by the number which stands in the fourth column of the said table, on the first line of each interval, placing the products under the signs +, or —, according to the signs prefix'd to the last mentioned numbers.

Multiply the number, which stands in the second column of the table of observations, on a line with the given age, by the interest of 1 l. placing the product under the sign +. Also multiply the last found product, again, by the interest of 1 l. reserving the product for future use.

Multiply the number which stands in the third column of the table of observations, on a line with the given age, by the rate; and place this product under the sign —.

Subtract the sum of all the numbers, which stand under the sign —, from the sum of all those which stand under the

SCHOLIUM.

If the result of the two last examples (whereof the first gives 16,3732 *l.* for the value of an annuity of 1 *l.* for a life of 10 years, according to the *London* bills; and the latter gives 17,7237 *l.* for the value of an annuity on the same life, according to the *Breslaw* bills) be compared with the value of the same annuity, supposing the decrements of life to be equal; which (by quest. 56) was found to be 16,8814; it appears that the result, according to that hypothesis, is greater than the former, and lesser than the latter; and consequently, for general use, may be more eligible than either.

As the table of the present worths of 1 *l.* due at the end of any number of years to come, is necessary to the solution of this and many other questions relating to annuities on lives; it is inserted here, that the reader may not have the inconvenience of turning to another book, for the same.

And because the values of single lives, computed upon the supposition of the equality of the decrements of life, are also necessary to the approximation to the values of such combined lives, a table containing them is also annexed.

A TABLE

A TABLE of the present Values of One Pound.

	3 per Ct.	$3\frac{1}{2}$ per Ct.	4 per Ct.	$4\frac{1}{2}$ per Ct.	5 per Ct.	6 per Ct.
1	970874	966184	961539	956938	952381	943396
2	942596	933511	924556	915732	907029	889996
3	915142	901943	888996	876297	863838	839619
4	888487	871442	854804	838561	822702	792094
5	862609	841973	821927	802451	783526	747258
6	837484	813501	790315	767896	746215	704961
7	813092	785991	759918	734828	710681	665057
8	789409	759412	730690	703185	676839	627412
9	766417	733731	702587	672904	644609	591898
10	744094	708919	675564	643928	613913	558395
11	722421	684946	649581	616199	584679	526788
12	701380	661783	624597	589664	556837	496969
13	680951	639404	600574	564272	530321	468839
14	661118	617782	577475	539973	505068	442301
15	641862	596891	555265	516720	481017	417265
16	623167	576706	533908	494469	458112	393646
17	605016	557204	513373	473176	436297	371364
18	587395	538361	492628	452800	415521	350344
19	570286	520156	474642	433302	395734	330513
20	553676	502566	456387	414643	376889	311805
21	537549	485571	438834	396787	358942	294155
22	521893	469151	421955	379701	341850	277505
23	506692	453286	405726	363350	325571	261797
24	491934	437957	390121	347703	310068	246979
25	477606	423147	375117	332731	295303	232999
26	463695	408838	360689	318402	281241	219810
27	450189	395012	346817	304691	267848	207368
28	437077	381654	333477	291571	255094	195630
29	424346	368748	320651	279015	242946	184557
30	411987	356278	308319	267000	231377	174110
31	399987	344230	296460	255502	220359	164255
32	388337	332590	285058	244500	209866	154957
33	377026	321343	274094	233971	199873	146186

A TABLE

A TABLE of the present Values of One Pound.

3 per Ct. 3½ per Ct. 4 per Ct. 4½ per Ct. 5 per Ct. 6 per Ct.					
34	366045	310476	263552	223896	190355
35	355383	299977	253415	214254	181290
36	345032	289833	243689	205028	172657
37	334983	280032	234297	196199	164436
38	325226	270562	225285	187750	156605
39	315754	261413	216621	179665	149148
40	306557	252572	208289	171929	142046
41	297628	244031	200278	164525	135282
42	288959	235779	192575	157440	128840
43	280543	227806	185168	150663	122704
44	272372	220102	178046	144173	116864
45	264439	212659	171198	137964	111297
46	256737	205468	164614	132023	105997
47	249259	198520	158283	126338	100949
48	241999	191846	152195	120898	96142
49	234956	185320	146341	115692	91564
50	228107	179053	140713	110710	87204
51	221463	172998	135301	105942	83051
52	215013	167148	130097	101380	79096
53	208750	161496	125093	97014	75330
54	202670	156035	120282	92837	71743
55	196767	150758	115656	88839	68326
56	191036	145660	111207	85013	65073
57	185472	140734	106930	81353	61974
58	180070	135975	102817	77849	59023
59	174825	131377	98863	74497	56212
60	169733	126934	95060	71289	53536
61	164789	122642	91404	68219	50986
62	159990	118495	87889	65281	48558
63	155330	114487	84508	62470	46246
64	150800	110616	81258	59780	44044
65	146413	106875	78133	57206	41946
66	142149	103261	75128	54742	39949

A TABLE of the present Values of One Pound.

	3 per Ct.	3½ per Ct.	4 per Ct.	4½ per Ct.	5 per Ct.	6 per Ct.
67	,138009	,099769	,072238	,052385	,038047	,020161
68	,133989	,096395	,069460	,050129	,036235	,019020
69	,130086	,093136	,066788	,047971	,034509	,017943
70	,126297	,089986	,064219	,045905	,032866	,016927
71	,122619	,086943	,061749	,043928	,031301	,015969
72	,119047	,084003	,059374	,042037	,029811	,015065
73	,115580	,081162	,057091	,040226	,028391	,014212
74	,112214	,078418	,054895	,038494	,027039	,013408
75	,108945	,075766	,052784	,036836	,025752	,012649
76	,105772	,073204	,050754	,035250	,024525	,011933
77	,102691	,070728	,048801	,033732	,023357	,011258
78	,099700	,068336	,046924	,032280	,022245	,010620
79	,096796	,066026	,045120	,030890	,021186	,010019
80	,093977	,063793	,043384	,029559	,020177	,009452
81	,091240	,061636	,041716	,028287	,019216	,008917
82	,088582	,059551	,040111	,027068	,018301	,008412
83	,086002	,057538	,038569	,025903	,017430	,007936
84	,082497	,055592	,037084	,024787	,016600	,007487
85	,081035	,053712	,035659	,023720	,015809	,007063
86	,078704	,051896	,034287	,022699	,015056	,006663
87	,076412	,050141	,032968	,021721	,014339	,006286
88	,074186	,048445	,031700	,020786	,013657	,005930
89	,072027	,046807	,030481	,019891	,013006	,005595
90	,069928	,045224	,029309	,019034	,012387	,005278
91	,067891	,043695	,028182	,018215	,011797	,004979
92	,065914	,042217	,027098	,017430	,011235	,004697
93	,063994	,040789	,026055	,016680	,010700	,004432
94	,062130	,039410	,025053	,015961	,010191	,004181
95	,060320	,038077	,024090	,015274	,009705	,003944
96	,058563	,036790	,023163	,014616	,009243	,003721
97	,056858	,035546	,022272	,013987	,008803	,003510
98	,055202	,034344	,021416	,013385	,008384	,003312
99	,053594	,033182	,020592	,012808	,007985	,003124
100	,052033	,032060	,019800	,012257	,007604	,002957

A TABLE

A TABLE of the present Values of an Annuity of One Pound, on a single Life, supposing the Decrements of Life to be equal.

Age	3 per Ct.	3½ per Ct.	4 per Ct.	4½ per Ct.	5 per Ct.	6 per Ct.
8	19,736	18,160	16,791	15,595	14,544	12,790
9	19,868	18,269	16,882	15,672	14,607	12,839
10	19,868	18,269	16,882	15,672	14,607	12,839
<hr/>						
11	19,736	18,160	16,791	15,595	14,544	12,790
12	19,604	18,049	16,698	15,517	14,480	12,741
13	19,469	17,937	16,604	15,437	14,412	12,691
14	19,331	17,823	16,508	15,356	14,342	12,639
15	19,192	17,707	16,410	15,273	14,271	12,586
16	19,050	17,588	16,311	15,189	14,197	12,532
17	18,905	17,467	16,209	15,102	14,123	12,476
18	18,759	17,344	16,105	15,015	14,047	12,419
19	18,610	17,220	15,999	14,923	13,970	12,361
20	18,458	17,093	15,891	14,831	13,891	12,301
<hr/>						
21	18,305	16,963	15,781	14,737	13,810	12,239
22	18,148	16,830	15,669	14,641	13,727	12,177
23	17,990	16,696	15,554	14,543	13,642	12,112
24	17,827	16,559	15,437	14,442	13,555	12,045
25	17,664	16,419	15,318	14,340	13,466	11,978
26	17,497	16,277	15,197	14,235	13,375	11,908
27	17,327	16,133	15,073	14,128	13,282	11,837
28	17,154	15,985	14,946	14,018	13,186	11,763
29	16,979	15,835	14,816	13,905	13,088	11,688
30	16,800	15,682	14,684	13,791	12,988	11,610
<hr/>						
31	16,620	15,526	14,549	13,673	12,885	11,530
32	16,436	15,367	14,411	13,555	12,780	11,449
33	16,248	15,204	14,270	13,430	12,673	11,365
34	16,057	15,039	14,126	13,304	12,562	11,278
35	15,864	14,871	13,979	13,175	12,449	11,189
36	15,666	14,699	13,829	13,044	12,333	11,098

A TABLE of the present Values of an Annuity of One Pound, on a single Life, supposing the Decrements of Life to be equal.

Age	3 per Ct.	3½ per Ct.	4 per Ct.	4½ per Ct.	5 per Ct.	6 per Ct.
37	15,465	14,524	13,676	12,909	12,214	11,003
38	15,260	14,345	13,519	12,771	12,091	10,907
39	15,053	14,163	13,359	12,630	11,966	10,807
40	14,842	13,978	13,196	12,485	11,837	10,704
41	14,626	13,789	13,028	12,337	11,705	10,599
42	14,407	13,596	12,858	12,185	11,570	10,490
43	14,185	13,399	12,683	12,029	11,431	10,378
44	13,958	13,199	12,504	11,870	11,288	10,263
45	13,728	12,993	12,322	11,707	11,142	10,144
46	13,493	12,784	12,135	11,540	10,992	10,021
47	13,254	12,571	11,944	11,368	10,837	9,895
48	13,012	12,354	11,748	11,192	10,679	9,765
49	12,764	12,131	11,548	11,012	10,515	9,630
50	12,511	11,904	11,344	10,827	10,348	9,492
51	12,255	11,673	11,135	10,638	10,176	9,349
52	11,994	11,437	10,921	10,443	9,999	9,201
53	11,729	11,195	10,702	10,243	9,817	9,049
54	11,457	10,950	10,478	10,039	9,630	8,891
55	11,183	10,698	10,248	9,829	9,437	8,729
56	10,902	10,443	10,014	9,614	9,239	8,561
57	10,616	10,181	9,773	9,393	9,036	8,387
58	10,325	9,913	9,527	9,166	8,826	8,208
59	10,029	9,640	9,275	8,933	8,611	8,023
60	9,727	9,361	9,017	8,694	8,389	7,831
61	9,419	9,076	8,753	8,449	8,161	7,633
62	9,107	8,786	8,482	8,197	7,926	7,428
63	8,787	8,488	8,205	7,938	7,684	7,216
64	8,462	8,185	7,921	7,672	7,435	6,997
65	8,132	7,875	7,631	7,399	7,179	6,770

A TABLE

TABLE of the present Values of an Annuity of One Pound, on a single Life, supposing the Decrements of Life to be equal.

Age	3 per Ct.	3½ per Ct.	4 per Ct.	4½ per Ct.	5 per Ct.	6 per Ct.
56	7,794	7,558	7,333	7,119	6,915	6,535
57	7,450	7,234	7,027	6,831	6,643	6,292
58	7,099	6,902	6,714	6,534	6,362	6,040
59	6,743	6,585	6,394	6,230	6,073	5,779
60	6,378	6,219	6,065	5,918	5,775	5,508
61	6,008	5,865	5,728	5,596	5,468	5,228
62	5,631	5,505	5,383	5,265	5,152	4,937
63	5,246	5,136	5,029	4,926	4,826	4,636
64	4,854	4,759	4,666	4,576	4,489	4,324
65	4,453	4,373	4,293	4,217	4,143	4,000
66	4,046	3,978	3,912	3,847	3,784	3,664
67	3,632	3,575	3,520	3,467	3,415	3,315
68	3,207	3,163	3,111	3,076	3,034	2,953
69	2,776	2,741	2,707	2,673	2,641	2,578

QUESTION LXII.

Supposing the decrements of life to be in a constant ratio; that is, supposing the number of chances for its continuance in being one year, to be to the number of chances for its failing in that time, as a to b ; the number of chances for its continuance the second year, to the number of chances for its failing, as aa to bb , &c. it is required to find v the present value of 1 l . for that life, allowing the purchaser compound interest?

SOLUTION.

Let r be the amount of 1 l . in one year. Then the probability of the given life's surviving the first year will be

$$\frac{a}{a+b};$$

That of surviving the second year

$$\frac{aa}{a+b^2};$$

That of the third

$$\frac{aaa}{a+b^3};$$

Which expressions, being taken as the values of each payment in the parts of 1 l . the sum of their present values will be the present value of the annuity, *viz.*

$$\frac{a}{a+bxr} + \frac{aa}{a+b^2xr^2} + \frac{aaa}{a+b^3xr^3} + \frac{a^4}{a+b^4xr^4}; \text{ \&c.}$$

which, being a geometrical progression, infinitely decreasing, whose greatest term is $\frac{a}{a+bxr}$, ratio $\frac{a+bxr}{a}$,

$$\frac{a}{a+bxr}$$

and

and the ratio less unity $\left(\frac{a+b \times r}{a} - 1 = \right) \frac{a+b \times r - a}{a}$,
 the sum thereof will be the quotient arising from the di-
 vision of $\frac{a}{a+b \times r} \times \frac{a+b \times r}{a}$ by $\frac{a+b \times r - a}{a}$: But
 $\frac{a}{a+b \times r} \times \frac{a+b \times r}{a} = \frac{1}{1} = 1$; And 1 divided by
 $\frac{a+b \times r - a}{a}$ will quote $\frac{a}{a+b \times r - a} (=N)$ the present
 value required, by quest. 169. part 2. vol. I.

E X A M P L E.

Suppose a life, the number of chances for the continu-
 ance of which for one year, is to the number of chances
 for its failing in that time, as 1 to 0,03735, and that
 the decrements of life are in a constant ratio, what is
 its present value, allowing 4 *per Cent* compound interest?

Here $a=1; b=0,03735; a+b=1,03735; \&r=1,04$:
 Then $a+b=1,03735$,
 $r=1,04$

$$a+b \times r = 1,07884$$

$$a+b \times r - a = 0,07884; \text{ And } N = \left(\frac{1}{0,07884} \right) 12,683.$$

Q U E S T I O N LXIII.

Supposing the decrements of life to be in a constant
 ratio: If the value of an annuity thereon be given, to-
 gether with the rate of compound interest, it is required

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to find the probability of that life's continuing for one year.

SOLUTION.

If the symbols be retained as in the last question;
Then $\frac{a}{a+b \times r - a} = N$; by the result thereof; and it is
required to find the value of $\frac{a}{a+b}$, in the terms of N ,
and r .

Now $a = (N \times \overline{a+b \times r - a}) = \overline{a+b \times r} N - a N$.

And $a + a N = \overline{a+b} \times r N$.

Th. $\frac{a + a N}{a + b} = r N$

And $\frac{a}{a+b} = \frac{r N}{1+N}$

EXAMPLE I.

If $N=13,683$ and $r=1,04$ as before, then, $r N =$
 $13,9032$ And $\left(\frac{13,9032}{13,683}\right) 0,963993 = \frac{a}{a+b}$

EXAMPLE II.

If $N=10,478$, and $r=1,04$:

Then $r N = 10,89712$ and $\frac{a}{a+b} = 0,9494$.

EX.

EXAMPLE III.

If $N=7,333$. and $r=1.04$:

Then $rN=7,62632$; and $\left(\frac{7,62632}{8,333}\right) \frac{a}{a+b} = 9151$.

QUESTION LXIV.

Supposing the detrements of life to be equal, the present value of an annuity of 1*l.* to continue during the joint lives of two persons, whose respective complements of life are m , and n (m being the greater number) is required.

SOLUTION.

Since the probability of the continuance of the life, whose complement is n , for one year, is $\frac{n-1}{n}$, per qu. 56.

and that of the life, whose comp. is m , $\frac{m-1}{m}$, per ditto;

also, since these two events are independent on each

other, it will follow (from quest. 28.) that $\frac{n-1}{n} \times \frac{m-1}{m}$

will be the probability of both of those lives continuing in being one year; and by reasoning in the same way

$\frac{n-2}{n} \times \frac{m-2}{m}$ will be the probability thereof the second

year; $\frac{n-3}{n} \times \frac{m-3}{m}$ the third, &c. Which probabilities,

being taken as the values of the several payments, which will become due at the end of the first, second, third,

I 4.

&c.

$\&c.$ years ; and their present worths being found, as before, it will appear, that the value of the annuity will be expressed by the following series,

$$\frac{n-1 \times m-1}{nmr} + \frac{n-2 \times m-2}{nmr^2} + \frac{n-3 \times m-3}{nmr^3}, \&c.$$

Of which series n terms, only, will be useful ; because the life, whose complement is n , is supposed to be necessarily extinct in n years.

Now if the numerators of the fractions, which constitute the above series, be expanded, by multiplication, they will appear as below,

$$\begin{aligned} n-1 \times m-1 &= (nm - m - n + 1) = nm - m + n \times 1 + 1, \\ n-2 \times m-2 &= (nm - 2m - 2n + 4) = nm - m + n \times 2 + 4, \\ n-3 \times m-3 &= (nm - 3m - 3n + 9) = nm - m + n \times 3 + 9, \\ \&c. & \qquad \qquad \&c. & \qquad \qquad \&c. \end{aligned}$$

And therefore the above series may be divided into three other series, *viz.*

$$\frac{nm}{nm} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} (n) = \frac{nm}{nm} \times \frac{1-p}{r-1} \text{ by quest. 15.}$$

(where p is the present worth of 1 *l.* due at the end of n years)

$$- \frac{m+n}{nm} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} (n) = - \frac{m+n}{nm} \times \frac{1-p \times r}{r-1} - \frac{np}{r-1},$$

(by quest. 16.)

$$+ \frac{1}{nm} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} (n) = \frac{1}{nm} \times \frac{1-p \cdot r + 1 \cdot r}{r-1} - \frac{2np}{r-1} - \frac{n^2 p}{r-1}.$$

(by quest. 17.)

The

The sums of which three series, being ranged according to their respective divisors and signs, will stand as below,

$$\begin{aligned} & \frac{nm}{nm} \times \frac{1-p}{r-1}, \\ & + \frac{m+n}{nm} \times \frac{np}{r-1} - \frac{m+n}{nm} \times \frac{1-p \times r}{r-1}, \\ & - \frac{1}{nm} \times \frac{np}{r-1} - \frac{1}{nm} \times \frac{2rnp}{r-1} + \frac{1}{nm} \times \frac{1-p \cdot r + 1 \cdot r}{r-1}. \end{aligned}$$

But $\frac{m+n}{nm} \times \frac{np}{r-1} - \frac{1}{nm} \times \frac{np}{r-1} = \frac{mn}{nm} \times \frac{p}{r-1},$

And $\frac{nm}{nm} \times \frac{1-p}{r-1} + \frac{mn}{nm} \times \frac{p}{r-1} = \frac{mn}{nm} \times \frac{1}{r-1} = \frac{1}{r-1};$

Also $\frac{m+n}{nm} \times \frac{1-p \times r}{r-1} + \frac{1}{nm} \times \frac{2rnp}{r-1} = \frac{m+n \times r - m - n \times r p}{nm \times r - 1}.$

Therefore the value of the annuity will be

$$\frac{1}{r-1} - \frac{m+n \times r - m - n \times r p}{nm \times r - 1} + \frac{1-p \times r + 1 \times r}{nm \times r - 1}.$$

And putting $\frac{1}{r-1} = P$ (the value of the perpetuity) it will be

$$P - \frac{m+n \times r - m - n \times r p}{nm \times r - 1} + \frac{1-p \times r + 1 \times r}{nm \times r - 1}; \text{ Or}$$

$$P - \frac{m+n \times r - m - n \times r p}{nm} \times \frac{rPP}{r-1} + \frac{1-p \times r + 1 \times r}{nm} \times \frac{rPP}{r-1};$$

$$\text{Or } P - \frac{m+n \times r - m - n \times r p - 1-p \times r + 1 \times r}{nm} \times \frac{rPP}{r-1}.$$

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Although this may be the readiest expression to compute by, yet, for some subsequent purposes, it may be better to express this value otherwise by putting $\frac{1}{r-1} = P$,

$\frac{r}{r-1} = Q$ and $\frac{r+r}{r-1} = R$ as in cor. quest. 19. And then

the value $\frac{1}{r-1} \frac{m+n \times r - m - n \times r p}{nm \times r - 1^2} + \frac{1-p \times r + 1 \times r}{nm \times r - 1^3}$;

will become $P \frac{m+n - m - n \times p}{nm} Q + \frac{1-p}{nm} R$,

Or $P + \frac{m - n + mp - np}{nm} Q + \frac{1-p}{nm} R$,

Or $P + Q \times \frac{1}{n} - \frac{1}{m} + \frac{p}{n} - \frac{p}{m} + R \times \frac{1}{nm} - \frac{p}{nm}$;

And (by restoring $\frac{1}{r^n}$ for p .)

$$P + Q \times \frac{1}{n} - \frac{1}{m} + \frac{1}{nr^n} - \frac{1}{mr^n} + \left(R \times \frac{1}{nm} - \frac{1}{nmr^n} \right)$$

EXAMPLE I.

What is the value of an annuity of 1 £ to continue during the joint lives of two persons, of the respective ages of 43 and 54, supposing the decrements of life to be equal, and that compound interest be allowed at 4 per Cent.

Here

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Here $n=(86-54=)32$; $m=(86-43=)43$; $p=0,285058$;

$$r=1,04; \text{ and } P = \frac{1}{0,04} = 25.$$

Then $p=0,285058$; $1-p=0,714942$; $m+n=75$

$$\begin{array}{r} m-n = 11 \quad r+1 = 2,04 \\ \hline 3,135638; \quad 1,458482; \quad -3,135638 \\ \hline P = 25 \quad -36,462050 \\ \hline PP = 625, \quad 36,462050 \text{ Rems. } 35,402312 \\ \hline mn = 1376 \quad r = 1,04 \\ \hline 36,818404 \end{array}$$

And $36,818404 \times \frac{625}{1376} = 16,723475$

Lastly $25 - 16,723475 = 8,276525$, the present worth of the annuity.

E X A M P L E II.

What is the present value of an annuity of 1*l.* to continue during the joint Lives of two persons, the complements of whose lives are severally 43 and 20, allowing 4*l. per Cent.* compound interest?

Here $m=43$; $n=20$; $p=0,456387$; $r=1,04$; $P=25$.

Then $p=0,456387$; $1-p=0,543613$; $m+n=63$

$$\begin{array}{r} m-n = 23 \quad r+1 = 2,04 \\ \hline 10,496901; \quad 1,108970; \quad -10,496901 \\ \hline P = 25 \quad -27,724250 \\ \hline PP = 625 \quad 27,724250 \text{ Rems. } 24,778849 \\ \hline mn = 860 \quad r = 1,04 \\ \hline 25,770003 \end{array}$$

And $25,770003 \times \frac{625}{860} = 18,7282$

Lastly $25 - 18,7282 = 6,2718$ the answer. E. X.

EXAMPLE III.

What is the present value of an annuity of 1 £ for the joint lives of two persons, whose complements of life are 32 and 20, allowing compound interest at 4 *l. per Cent.*

Here $m=32$; $n=20$; $p=0.456387$; $r=1.04$; $P=25$.

Then $p=0.456387$; $1-p=0.543613$; $m+n=52$.

$$\begin{array}{r}
 m-n=12 \quad r+1 \quad 2.04 \\
 \hline
 5,476644 \quad 1,108970 \quad -5,476644 \\
 \hline
 P=25 \quad -27,724250 \\
 \hline
 PP=625 \quad 27,724250 \quad \text{Rems. } 18,799106; \\
 \hline
 mn=640 \quad r=1.04 \\
 \hline
 19,551070;
 \end{array}$$

$$\text{And } 19,551073 \times \frac{625}{640} = 19,092842$$

Lastly $25 - 19,092842 = 5,907158$
the value of the annuity required.

S C H O L I U M.

The value of the above annuity may be obtained, nearly, by a shorter process, if the present worth of an annuity on that single life, whose complement is n , be a known number, as follows:

$$\text{The series } \frac{n-1 \cdot m-1}{nmr} + \frac{n-2 \cdot m-2}{nmr^2} + \frac{n-3 \cdot m-3}{nmr^3} (n),$$

which expresses the value of the annuity, may be considered as composed by multiplying the corresponding terms of the two following series, together.

$$\frac{m-1}{m} + \frac{m-2}{m} + \frac{m-3}{m} + \frac{m-4}{m} (n),$$

$$\text{And } \frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} + \frac{n-4}{nr^4} (n);$$

The first of which is an arithmetical progression, whose greatest term is $\frac{m-1}{m}$, common difference $\frac{1}{m}$, and number of terms n ; Therefore (by question 17, part 2. vol. I.) the sum thereof is $\left(\frac{n \times m - 1}{m} - \frac{n \cdot n - 1}{2 \cdot m} = \right)$

$$\frac{2n \times m - 1 - n \times n - 1}{2m},$$

$$\text{Or } \left(\frac{2mn - 2n - nn + n}{2m} = \frac{2mn - nn - n}{2m} = \right) n - \frac{n \cdot n + 1}{2m}$$

And, although the second series (being the value of that single life whose complement is n) is not strictly speaking an arithmetical progression, yet as its sum is given $= N$; and as r , the ratio of the geometrical progression of divisors, is not much greater than unity, it may be considered as nearly such; and the common difference may be found per quest. 6. part 2. vol. I. for the greatest term is $\frac{n-1}{nr}$; and the least term is $\frac{n-n}{nr^n} = 0$; whose difference is $\frac{n-1}{nr}$; which, being divided by $n-1$, quotes $\frac{1}{nr}$, for the common difference thereof.

Now, by quest. 21. the sum of n terms of such a series of products may be found from the above data,

$$= N$$

$$= \frac{N}{n} \times \frac{n+1 \cdot n}{2m} + \frac{n+1 \cdot n \cdot n-1}{3 \cdot 2 \cdot 2} \times \frac{1}{m} \times \frac{1}{m}$$

$$\text{Or } N \times 1 - \frac{n+1}{2m} + \frac{n+1 \cdot n-1}{12mr}$$

Where, if we put $\frac{n+1}{2m} = q$; it will be

$$N - Nq + \frac{n-1}{6r} q \text{ or } N - Nq - \frac{n-1}{6r} q$$

$$\text{That is } N - N - \frac{n-1}{6r} \times q$$

EXAMPLE I.

What is the value of an annuity of 1 £. to continue during the joint lives of two persons, of the respective ages of 43 and 54. See example I. *aforegoing*.

Here $m=43$; $n=32$; $r=1.04$; and $N=10,478$, by Example 2, quest. 56.

Then $n-1=31$; $6r=6.24$; $n+1=33$; and $2m=86$;
 $6(32)31,000 (4,9679$, and $86) 33,000 (.38372=q$

$$N=10,478$$

$$- 4.968$$

$$\frac{N=10,478}{- 4.968}$$

$$\text{And } 5,510 \times 0.38372 = 2,114$$

Remains 8,364 the answer.

Which answer differs from the true answer, before found, only by being 0,088 too much.

EXAMPLE II.

What is the value of two joint lives, whose complements are 43 and 20 £. See example II. *aforegoing*.

Here

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Here $m=43$; $n=20$; $r=1,04$; and $N=7,333$, by example 3. quest. 56.

Now $n-1=19$; $6r=6,24$; $n+1=21$ and $2m=86$,
6,24) 19,000 (3,045, and 86) 21,000 (2,4418;

$N=7,333$

—3,045

$N=7,333$

And $4,288 \times 0,24418=1,047$

Remains 6,286 the answer.

Which is but 0,014 greater than the true answer.

E X A M P L E III.

What is the value of two joint lives, whose complements are 32 and 20? See example III. *abovegoing*.

Here $m=32$; $n=20$; $r=1,04$; $N=7,333$;

Now $n-1=19$; $6r=6,24$; $n+1=21$; and $2m=64$,
6,24) 19,000 (3,045, and 64) 21,000 (3,28125;

$N=7,333$

—3,045

$N=7,333$

$4,288 \times 0,328125=1,407$

5,926 the answer.

Which is but 0,019 greater than the true answer; and none of the above differences are $\frac{1}{10}$ of a year's purchase.

Since the answers, obtained by this approximation, differ so little from the true answers; and since the process is so much shorter; we may safely, in common cases, make use thereof, for which reason, it is here annexed in words at length.

The

The Rule to find the present value of an annuity, which is to continue during the joint lives of two persons of given ages; allowing compound interest at a given rate, supposing the decrements of life to be equal, and having the value of the oldest life given.

Let the number of years, which each of the persons want of 86, be called their complements of life; and let the sum of 1 l. and its interest for 1 year, be called the rate.

From the lesser complement subtract one, and divide the remainder by six times the rate; or find this quotient, in table the last.

From the value of the life of the oldest person subtract the above found quotient, and multiply the remainder by the lesser complement more one.

Divide the last found product by twice the greater complement, and subtract the quotient from the value of the oldest life, then the remainder will be the value of an annuity for the joint lives, which was required.

E X A M P L E.

What is the value of an annuity of 1 l. to continue during the joint lives of two persons of the respective ages of 43 and 54; allowing compound interest at four per Cent. the value of the life of 54 years being 10,478?

Here $(86 - 43 =) 43$, is the greater complement.

And $(86 - 54 =) 32$, is the lesser complement.

Also $(1 + .04 =) 1.04$ is the rate.

Now if $(32 - 1 =) 31$ be divided by $(1.04 \times 6 =) 6.24$, the quotient will be 4,968.

From 10,478, take 4,968, and there will remain 5,510.

Which multiplied by $(32 + 1 =) 33$ will produce 181,830.

If the last product 181,830, be divided by $(43 \times 2 =) 86$, the quotient will be 2,114.

Which quotient, taken from 10,478, will leave 8,364 for the value of the annuity required.

QUESTION LXV.

What is the present value of an annuity of 1 *l.* to continue during the joint lives of two persons, each aged 48 years, supposing the decrements of life to be equal and that compound interest be allowed at 4 *per Cent.*

SOLUTION.

The solution of this question may be deduced from that of quest. 64; for if the two complements of life m and n be equal, it will follow that, $m + n = 2n$; $m - n \times p = 0$; and $nm = nn$: Therefore the

value of the annuity will be $\frac{1}{r-1} - \frac{2r}{n \times r-1}$

$$+ \frac{1-p \times r+1 \times r}{nn \times r-1},$$

That is $P - \frac{2n-1-p \times r+1 \times P \times rPP}{nn}$:

Or in order to shew its connexion } $P - \frac{2}{n} Q + \frac{1-p}{nn} R.$
with some following solutions

In this case,

$$n = (86 - 48) = 38; r = 1.04; p = 0.225285; \text{ and } P = 25.$$

$$1 - p = .774715; 2n = 76, \quad PP = 625,$$

$$r + 1 = \frac{2.04}{-39.510475} \quad nn = 1444;$$

$$P = \frac{1,580,419}{25} \text{ Rems. } 36,489,525, \quad r = \frac{1.04}{1.04}$$

$$\frac{39,510,475}{1444} \text{ Now } 37,949,106 \times \frac{625}{1444} = 16,425,34$$

$$\text{And } 25 - 16,425,34 = 8,574,66, \text{ the value required.}$$

SCHOLIUM. But if N , the value of the single life whose complement is n , be given; then, by the scholium to the last question, the value of the annuity will be nearly,

$$N \times 1 - \frac{n+1}{2n} + \frac{n+1 \times n-1}{12nr}; \text{ Or, because } 1 - \frac{n+1}{2n} = \frac{n-1}{2n},$$

$$\left(N \times \frac{n-1}{2n} + \frac{n+1 \times n-1}{12nr} \right) = N + \frac{n+1}{6r} \times \frac{n-1}{2n}.$$

Now by observing the operations of the last question, it appears that the results, obtained by the above approximation, are greater than the truth: Therefore, when n is not a very small number, we may safely write

$$\left(\frac{n+1 \times n-1}{12r \times n+1} \right) = \frac{n-1}{12r}, \text{ for } \frac{n+1 \times n-1}{12rn}; \text{ because, there-}$$

by, the quotient will be diminished a little; and then the annuity will be worth, nearly,

$$\left(N \times \frac{n-1}{2n} + \frac{n-1}{12r} \right) = \frac{N}{n} + \frac{1}{6r} \times \frac{n-1}{2}.$$

Which, in words at length, follows:

The Rule for finding the present value of an annuity, to continue during the joint lives of two persons of equal ages, having the value of the single life given, supposing the decrements of life to be equal, and allowing compound interest at a given rate.

Let the number of years, which the given age wants of 86, be called the complement of life; and let the sum of one pound, and its interest for one year, be called the rate.

Divide the given value of the single life, by the complement of life; also divide unity by six times the rate, (which last quotient is in the first line of table the last.)

Add those two quotients together, and multiply their sum by the complement less one; then shall half the product be the value of the annuity required.

E X A M P L E I.

If the given age of the two persons be 48 years, and the rate of interest four per Cent. then the value of the single life will be 11,748.

Then $(86 - 48 =) 38$ will be the complement of life,

And $(1 + .04 =) 1.04$ will be the rate.

Then, if 11,748 be divided by 38, the quotient will be 0,30917, and if one be divided by $(6 \times 1.04 =) 6.24$, the quotient will be 0,16026.

Also if 0,30917 be added to 0,16026, and the sum 0,46943 be multiplied by $(38 - 1 =) 37$ the product will be 17,3630; the half of which, viz. 8,6815, is the value of the annuity required.

E X A M P L E II.

If the two persons be each 54 years of age, allowing compound interest at the same rate?

Then

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Then 10,478, will be the value of the single life.

(86—54=) 32, the complement of life :

And (1+,04=) 1,04, the rate.

Now if 10,478, be divided by 32, the quotient will be
0,3275.

Then (0,3275+0,1603=) 0,4878 being multiplied
by (32—1=) 31, produces 15,1218; the half of which,
viz. 7,5609, is the value of the annuity required.

QUESTION LXVI.

Supposing the decrements of life to be in a constant ratio, let there be two lives, the probabilities of the continuance of which for one year are respectively x and y ; it is required to find the present value of an annuity on their joint continuance, allowing compound interest to the purchaser?

SOLUTION.

By the principles above given, the probability of both the lives continuing the first year will be xy , the second year xyx , the third year x^2y^2 , &c.

Which expressions being the value of each payment, their present worths $\left(\frac{xy}{r} + \frac{xyx}{rr} + \frac{x^2y^2}{r^2}, \text{ \&c.}\right)$ will be the value of the annuity required.

Which is a geometrical progression infinitely decreasing for x and y are both less than unity) whose greatest term is $\frac{xy}{r}$; ratio $\frac{r}{xy}$; and ratio less unity $\left(\frac{r}{xy} - 1 =\right)$

$r - xy$

$\frac{r-xy}{xy}$; Therefore the sum thereof will be $\frac{xy}{r-xy}$; which is the value of the annuity required.

QUESTION LXVII.

Having given the values of the annuities of two single lives (computed at a known rate, *per Cent.*) to find the value of an annuity for their joint lives, allowing compound interest at the same rate, and supposing the decrements of life to be in a constant ratio?

SOLUTION.

Let the given values of the two annuities be N and M ; and r the amount of 1 £ in one year.

Then, (since the decrements of life are in a constant ratio) the probabilities of the continuance of those lives each for one year will be respectively $\frac{rN}{1+N}$ and $\frac{rM}{1+M}$ by question 63, which probabilities being substituted, in the places of x and y , in the result of question 66, (*viz.*

$\frac{xy}{r-xy}$) it will then become, the quotient of $\frac{rN}{1+N} \times$

$\frac{rM}{1+M}$ divided by $\left(r - \frac{rN}{1+N} \times \frac{rM}{1+M} = \right)$

$$\frac{\frac{rN}{1+N} \times \frac{rM}{1+M}}{1+N \times 1+M \times r - rN \times rM};$$

That is $\frac{rN}{1+N} \times \frac{rM}{1+M} \times \frac{1+N \times 1+M}{1+N \times 1+M \times r - rN \times rM}$

will be the value of the annuity required.

Which

Which reduced becomes $\frac{rNM}{1+N \times 1+M-rNM}$;

Which is the theorem given by the celebrated Mr. De Moivre, in the first edition of his treatise of annuities on lives; who argues thus:

“ Although the decrements of life be not really in a constant ratio; yet as the values of the two single lives are determined, the combinations of two or more such lives will be, nearly, the same with the combinations of two or more lives whose decrements are in a constant ratio.”

EXAMPLES.

I. What is the value of an annuity, on two joint lives, which singly computed (at 4 *l. per Cent. per Annum*) are respectively worth 12,683 and 10,478?

$$\text{Here } 12,683 \times 10,478 = 157,053$$

$$\text{And } 12,683 \times 10,478 \times 1,04 = 138,208$$

$$18,845 \overline{) 138,208} (7,334$$

II. What is the value of an annuity on two joint lives, which singly computed (at 4 *l. per Cent. per Annum*) are respectively worth 12,683 and 7,333?

$$\text{Here } 12,683 \times 7,333 = 114,020$$

$$\text{And } 12,683 \times 7,333 \times 1,04 = 96,625$$

$$17,395 \overline{) 96,625} (5,554$$

III. What is the value of an annuity on two joint lives, which singly computed (at 4 *l. per Cent. per Ann.*) are respectively worth 10,478 and 7,333?

Here

Here $11,478 \times 8,333 = 95,746$

And $10,478 \times 7,333 \times 1,04 = 79,909$

$15,837) 77,909 (5.045$

S C H O L I U M.

If these results be compared with the answers, upon the principle of equal decrements of life (found by quest. 64.) the values of the single lives, given in the above examples, being the same with the values of the single lives of the ages therein mentioned, it will appear that this rule will give the values of joint lives considerably less than that.

For the value of 2 lives of the ages 43 and 54 is } 8,276;

The same by the above - - - - - 7,334;

Difference - - - - - 0,942.

The value of 2 lives of the ages 43 and 66 is 6,272;

The same by the above - - - - - 5,554;

Difference - - - - - 0,718.

The value of 2 lives of the ages 54 and 66 is 5,907;

The same by the above - - - - - 5,045;

Difference - - - - - 0,862.

Which differences at a medium are about $\frac{1}{3}$ of a year's purchase.

COROL,

COROL.

If the lives are of equal ages, then the value of the annuity on their joint continuance will be

$$\frac{rNN}{1+N^2-rNN}$$

QUESTION LXVIII.

The present value of annuity of 1 *l.* to continue during the joint lives of two persons, of the respective ages of 10 and 31 (at 4 *per Cent.*) according to the table of observations, deduced from the bills of mortality, of *London*, is required?

SOLUTION.

If the probabilities of the continuance of the respective two lives, for one, two, three, &c. years, be taken from the said tables of observations, they will be as follow, *viz.*

For the life of 10 years $\frac{517}{524}, \frac{510}{524}, \frac{504}{524}, \&c.$

For the life of 31 years $\frac{367}{376}, \frac{358}{376}, \frac{349}{376}, \&c.$

And consequently the probability of their joint continuance for one, two, three, &c. years, will be

$\frac{517 \times 367}{524 \times 376}, \frac{510 \times 358}{524 \times 376}, \frac{504 \times 349}{524 \times 376}$, the present worths of which, *viz.*

$$\frac{517 \times 367}{524 \times 376 \times 1.04} + \frac{510 \times 538}{524 \times 376 \times 1.04^2} + \frac{504 \times 349}{524 \times 376 \times 1.04^3}$$

(63) will be the value of the annuity required.

However tedious the numerical operation, (deduced from the above) may appear, no demonstrated method that will shorten it has been published: for which reason, it has been thought proper to compare the result thereof with the two approximations above given.

As this series consists of fractions, having two factors in the numerator, and three in the denominator (whereof two are invariable), the value of each fraction will be easily obtained by a logarithmic process; previous to which, it will be necessary to find the logarithm of the product of the two invariable factors: thus,

$$\text{Log. of } 524 = 2.7193$$

$$\text{Log. of } 376 = 2.5752$$

$$\text{Log. of } 524 \times 376 = 5.2945$$

The No. to the last Log. being the values of the Fractions.	0.9250 0.8568 0.7936 0.7347 0.6794 0.6278 0.5794 0.5344 0.4906 0.4499 0.4116 0.3749 0.3428 0.3129 0.2845 0.2583
The diff. of the Log of the Nu. and Den. being the Log. of the Fractions.	1.9667 1.9329 1.8996 1.8661 1.8321 1.7978 1.7630 1.7279 1.6907 1.6531 1.6139 1.5740 1.5350 1.4954 1.4541 1.4121
The last added to 5.2945, being the Log. of the Denominators.	5.3115 5.3286 5.3456 5.3627 5.3797 5.3967 5.4138 5.4308 5.4478 5.4649 5.4819 5.4989 5.5160 5.5330 5.5500 5.5671
Logarithms of the powers of ($r=$) 1.04.	0.0170 0.0341 0.0511 0.0681 0.0852 0.1022 0.1192 0.1363 0.1533 0.1703 0.1874 0.2044 0.2214 0.2384 0.2555 0.2725
Sums of those Log. being the Log. of the Nu- merators.	5.2782 5.2615 5.2452 5.2287 5.2118 5.1945 5.1767 5.1587 5.1385 5.1179 5.0958 5.0729 5.0509 5.0283 5.0041 4.9791
Logarithms of those Num- bers.	2.5647 2.5339 2.5428 2.5315 2.5198 2.5079 2.4955 2.4829 2.4683 2.4533 2.4378 2.4216 2.4065 2.3909 2.3747 2.3579
No. taken from Tab. Observ.	367 358 349 340 331 322 313 304 294 284 274 264 255 246 237 228
Logarithms of those Num- bers.	2.7135 2.7076 2.7024 2.6972 2.6920 2.6866 2.6812 2.6758 2.6702 2.6646 2.6580 2.6513 2.6444 2.6374 2.6294 2.6212
No. taken from Tab. Observ.	517 510 504 498 492 486 480 474 468 462 455 448 441 434 426 418

410	2, 6128	220	2, 3424	4, 9552	0, 2896	5, 5847	1, 3711	0, 2350
402	2, 6042	212	2, 3263	4, 9305	0, 3065	5, 6011	1, 3295	0, 2135
394	2, 5955	204	2, 3096	4, 9051	0, 3236	5, 6182	1, 2870	0, 1936
385	2, 5855	196	2, 2923	4, 8778	0, 3407	5, 6352	1, 2426	0, 1748
376	2, 5752	188	2, 2742	4, 8494	0, 3577	5, 6522	1, 1972	0, 1575
367	2, 5647	180	2, 2553	4, 8200	0, 3747	5, 6693	1, 1508	0, 1415
358	2, 5539	172	2, 2355	4, 7894	0, 3918	5, 6863	1, 1032	0, 1268
349	2, 5428	165	2, 2175	4, 7603	0, 4088	5, 7033	1, 0570	0, 1140
340	2, 5315	158	2, 1987	4, 7302	0, 4258	5, 7204	2, 0099	0, 1023
331	2, 5198	151	2, 1790	4, 6988	0, 4429	5, 7374	2, 9615	0, 0915
322	2, 5079	144	2, 1584	4, 6663	0, 4599	5, 7544	2, 9119	0, 0816
313	2, 4955	137	2, 1367	4, 6322	0, 4769	5, 7715	2, 8608	0, 0726
304	2, 4829	130	2, 1139	4, 5968	0, 4940	5, 7885	2, 8083	0, 0643
294	2, 4683	123	2, 0899	4, 5582	0, 5110	5, 8055	2, 7527	0, 0565
284	2, 4533	117	2, 0682	4, 5215	0, 5280	5, 8226	2, 6990	0, 0500
274	2, 4378	111	2, 0453	4, 4831	0, 5450	5, 8396	2, 6435	0, 0440
264	2, 4216	105	2, 0212	4, 4428	0, 5621	5, 8566	2, 5862	0, 0386
255	2, 4065	99	1, 9956	4, 4021	0, 5791	5, 8737	2, 5285	0, 0338
246	2, 3909	93	1, 9685	4, 3594	0, 5962	5, 8906	2, 4687	0, 0294
237	2, 3747	87	1, 9395	4, 3142	0, 6132	5, 9077	2, 4065	0, 0255

Sum of 36 terms of the series carried forwards—10,7038

The No. to the last Log. being the values of the fractions	0, 0220 0, 0189 0, 0161 0, 0138 0, 0118 0, 0099 0, 0083 0, 0070 0, 0059 0, 0050 0, 0042 0, 0036 0, 0030 0, 0024 0, 0019 0, 0015
The diff. of the Log of the Nu. and Den. being the Log. of the fractions.	2, 3417 2, 2757 2, 2063 2, 1400 2, 0703 3, 9967 3, 9186 3, 8447 3, 7693 3, 7005 3, 6281 3, 5515 3, 4700 3, 3827 3, 2884 3, 1890
The last added to 5,2945, being the Log. of the Denominators.	5, 9248 5, 9418 5, 9588 5, 9759 5, 9929 6, 0099 6, 0270 6, 0440 6, 0610 6, 0781 6, 0951 6, 1121 6, 1292 6, 1462 6, 1632 6, 1803
Logarithms of the powers of (=) 1,04.	0, 6302 0, 6473 0, 6643 0, 6813 0, 6984 0, 7154 0, 7324 0, 7495 0, 7665 0, 7835 0, 8006 0, 8176 0, 8346 0, 8516 0, 8687 0, 8857
Sums of those Log. being the Log. of the Nu- merators.	4, 2664 4, 2175 4, 1651 4, 1158 4, 0632 4, 0066 3, 9455 3, 8887 3, 8303 3, 7785 3, 7231 3, 6636 3, 5991 3, 5289 3, 4516 3, 3692
Logarithms of those Num- bers.	1, 9085 1, 8751 1, 8388 1, 8062 1, 7709 1, 7324 1, 6902 1, 6532 1, 6128 1, 5798 1, 5441 1, 5052 1, 4624 1, 4150 1, 3617 1, 3010
No. taken from Tab. Observ.	81 75 69 64 59 54 49 45 41 38 35 32 29 26 23 20
Logarithms of these Num- bers.	2, 3579 2, 3424 2, 3263 2, 3096 2, 2923 2, 2742 2, 2553 2, 2355 2, 2175 2, 1987 2, 1790 2, 1584 2, 1367 2, 1139 2, 0899 2, 0682
No. taken from Tab. Observ.	228 220 212 204 196 188 180 172 165 158 151 144 137 130 123 117

111	2,0453	17	1,2304	3,2757	0,9028	6,1973	3,0785	0,0012
105	2,0212	14	1,1461	3,1673	0,9198	6,2143	4,9510	0,0009
99	1,9956	12	1,0792	3,0748	0,9368	6,2314	4,8435	0,0007
93	1,9685	10	1,0000	2,9685	0,9539	6,2484	4,7201	0,0005
87	1,9395	8	0,9031	2,8426	0,9709	6,2654	4,5772	0,0004
81	1,9085	6	0,7782	2,6867	0,9879	6,2825	4,4043	0,0003
75	1,8751	5	0,6990	2,5741	1,0050	6,2995	4,2746	0,0002
69	1,8388	4	0,6021	2,4409	1,0220	6,3165	4,1245	0,0001
64	1,8062	3	0,4771	2,2833	1,0390	6,3336	5,9498	0,0001
59	1,7709	2	0,3010	2,0719	1,0561	6,3506	5,7213	0,0001
54	1,7324	1	0,0000	1,7324	1,0731	6,3676	5,3648	0,0000

Sum of the last 27 terms of the series — — — 0,1398

Sum of the first 36 ditto — — — 10,7038

Value of the annuity — 10,8436

Note, The Indexes of the last column of logarithms are negative, and were mark'd as such in the copy; but the author, being at a great distance from the press, hopes this advertisement to the reader will excuse the printer's omission, who must otherwise have alter'd the measure of his page, by which errors of greater consequence might have happened, or the publication have been retarded.

Mr. *Simpson* has obliged the world with tables, of the values of lives calculated from the *London* observations, in which the value of the single lives of the ages of 10 and 31 (at 4 *per Cent.*) are severally 16,4, and 12,9; which being used for the symbols *M* and *N* in quest. 67. will give Mr. *De Moivre's* approximation to the values of the above joint lives, *viz.*

$$\begin{array}{rcl} 17,4 \times 13,9 & = & 241,86 \\ 16,4 \times 12,9 \times 1,04 & = & 220,02 \end{array}$$

21,84) 220,02 (10,07 the value of the annuity; which, if compared with the above, is above $\frac{1}{2}$ of a year's purchase too little.

Again, if 12,9 be used for *N*, in the rule given in the scholium to question 64; that approximation to the value of the joint lives may stand thus.

$$(86 - 10 =) 76 = m; (86 - 31 =) 55 = n; \text{ And } (1, + 0,04 =) 1,04 = r \quad 1,04 \times 6 = 6,24; \text{ And } 6,24) 54,00 (8,6; \text{ then } 12,9 - 8,6 = 4,3;$$

$$\text{Also } (55 + 1 =) 56 \times 4,3 = 240,8; \text{ and } (2 \times 76 =) 152; \text{ Then } 152) 240,8 (1,6.$$

Lastly $12,9 - 1,6 = 11,3$, the value of the annuity; which is $\frac{1}{2}$ a year's purchase too much.

SCHOLIUM.

It will be very easy to account for the former approximation being nearer to, and the latter farther from, the truth, in this case, than when the decrements of life were supposed equal; for the former, which is founded on the number of the living being in a geometrical progression, is now better adapted to the numbers in the tables of observations (which the farther they decline from

from being truly arithmetical, the nearer they will necessarily approach to the being geometrical): than it was when the number of the living were supposed to be an arithmetical progression: And on the contrary, the latter approximation (being founded on the summation of the series of products, made by the separate terms of two arithmetical progressions) must necessarily be farther from the truth, when neither of those progressions is strictly arithmetical, than when but one of them was otherwise.

However, when we consider, in the question before us, that the life of 31 is truly found by the table of observations, and that the numbers used in the life of 10 (which may be supposed to be as young as will commonly occur in practice) contain, greater deviations from an arithmetical progression, than any elder age can; we may conclude, that (when the value of the younger life is computed from the tables) this approximation will never exceed the truth, by more than $\frac{1}{2}$ a year's purchase.

And if we farther consider, that the bills of mortality of *London* make the value of a single life lesser than the other methods of computation do; we may conclude, that such excess will not exceed the real value of the joint lives of such persons as live out of that metropolis; or even in it, if they live temperately.

Mr. *Simpson*, has given a very easy approximation (by the help of the tables of equal joint lives, inserted in his treatise) which in the solution of this example comes extremely near the truth; but as he has thought proper to conceal the means he used to attain it, and as it would take up a great deal of time and labour to try it in other instances, I am, much against my will, obliged to omit it, together with many

others (therein contained) which if true, are equally curious and useful: And the very same thing happens, with regard to the rule given for this purpose, by Mr. *De Moivre*, in the second and third editions of his treatise of annuities on lives; which, he therein says, was originally derived from that above given in quest. 67: As I have the honour to be intimately acquainted with him, I desired that he would be so good to give me the investigation thereof, in order to have inserted it; which he readily promised to do; but has since told me, that he cannot find it among his papers, and that he doth not retain the manner of doing it, in his memory.

QUESTION LXIX.

Supposing the decrements of life to be equal, the present value of an annuity of 1 l. to continue during the joint lives of three persons, whose respective complements of life are t , m , and n (t being $\square m$ and $m \square n$) is required?

SOLUTION.

Since the probability of the continuance of the life, whose complement is n , for one year is $\frac{n-1}{n}$ per quest. 56,

That of the life, whose complement is m , $\frac{m-1}{m}$ per ditto,

That of the life, whose complement is t , $\frac{t-1}{t}$ per ditto;

And, since these three events are independent on each other, it will follow, that $\frac{n-1}{n} \times \frac{m-1}{m} \times \frac{t-1}{t}$ will be the

pro-

probability of all the three lives continuing one year ;
and by a like manner of reasoning $\frac{n-2}{n} \times \frac{m-2}{m} \times \frac{t-2}{t}$
will be the probability of their continuing the second year ;
 $\frac{n-3}{n} \times \frac{m-3}{m} \times \frac{t-3}{t}$ the third year, &c.

Which probabilities, being taken as the values of the
several payments, which will become due at the end of
the first, second, third, &c. years; and their present
worths being found, as in quest. 56, it will appear that
the value of the annuity may be expressed by

$$\frac{n-1 \cdot m-1 \cdot t-1}{nmtr} + \frac{n-2 \cdot m-2 \cdot t-2}{nmtr^2} + \frac{n-3 \cdot m-3 \cdot t-3}{nmtr^3}, \text{ \&c.}$$

Of which series, n terms, only, will be useful ; be-
cause the life, whose complement is n , is supposed to be
necessarily extinct in n years.

Now, if the numerators of the terms of the above
series be expanded, by the actual multiplication of their
several factors, they will become, viz.

$$\begin{aligned} n-1 \cdot m-1 \cdot t-1 &= nm t - nm + nt + mt \times 1 + n + m + t \times 1 - 1, \\ n-2 \cdot m-2 \cdot t-2 &= nm t - nm + nt + mt \times 2 + n + m + t \times 4 - 8, \\ n-3 \cdot m-3 \cdot t-3 &= nm t - nm + nt + mt \times 3 + n + m + t \times 9 - 27, \\ &\text{ \&c.} \qquad \qquad \qquad \text{ \&c.} \end{aligned}$$

And therefore the above series may be divided into 4
other series, viz.

$$\begin{aligned}
& \frac{nm}{nm} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} (n), \\
& - \frac{nm+nt+mt}{nm} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} (n), \\
& + \frac{n+m+t}{nm} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4} (n), \\
& \text{And } - \frac{1}{nm} \times \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} (n);
\end{aligned}$$

Which series, being summ'd by question 15, 16, 17, and 18, severally become

$$\begin{aligned}
& \frac{nm}{nm} \times \frac{1-p}{r-1}, \text{ (where } p \text{ is the present worth of } 1 \text{ l. due at the end of } n \text{ years.)} \\
& - \frac{nm+nt+mt}{nm} \times \frac{1-p \times r}{r-1} - \frac{np}{r-1}, \\
& + \frac{n+m+t}{nm} \times \frac{1-p \times r + 1 \times r}{r-1} - \frac{2nrp}{r-1} - \frac{np}{r-1}, \\
& - \frac{1}{nm} \times \frac{1-p \times r + 2 \cdot r - 3r}{r-1} - \frac{r+1 \times 3nrp}{r-1} - \frac{3n^2rp}{r-1} - \frac{n^3p}{r-1};
\end{aligned}$$

Which sums, being ranged according to their respective divisors and signs may stand as follows.

$$\frac{nm}{nm}$$

$$\begin{aligned}
 & \frac{nm}{nmt} \times \frac{1-p}{r-1}, \\
 & + \frac{nm+nt+mt}{nmt} \times \frac{np}{r-1} - \frac{nm+nt+mt}{nmt} \times \frac{1-p \times r}{r-1^2}, \\
 & - \frac{n+m+t}{nmt} \times \frac{np}{r-1} - \frac{n+m+t}{nmt} \times \frac{2np}{r-1^2} + \\
 & + \frac{1}{nmt} \times \frac{n^2 p}{r-1} + \frac{1}{nmt} \times \frac{3n^2 p}{r-1^2} + \\
 & \left(\frac{n+m+t}{nmt} \times \frac{1-p \times r+1 \times r}{r-1^3} \right. \\
 & \left. \left(\frac{1}{nmt} \times \frac{r+1 \times 3np}{r-1^3} - \frac{1}{nmt} \times \frac{1-p \times r+2^2 \times r-3r}{r-1^4} \right) \right)
 \end{aligned}$$

To consider these in their order, let us begin with the numerators of these four fractions, whose common denominator is $nmt \times r-1$:

$$\begin{aligned}
 & + nm \times 1 - p - nm - nmp, \\
 & + nm + nt + mt \times np = nm p + n^2 p + m^2 p, \\
 & - n + m + t \times np = -n^2 p - n^2 t p - n^2 p, \\
 \text{And } & + 1 \times n^3 p = + n^3 p.
 \end{aligned}$$

The sum of which is nmt ;

Th. $\left(\frac{nm}{nmt} \times \frac{1}{r-1} \right) \frac{1}{r-1}$ will be the value of these four fractions.

Let us next consider these three fractions, whose common denominator is $nmt \times r-1^2$;

Here, if we put $nm+nt+mt=K$,

$$K - S \times r - rp = -Kr + nmrp + ntrp + mtrp;$$

$$-n+m+t \times 2rnp = -2nmr - 2ntr - 2rn^2p;$$

$$\text{And } +1 \times 3rn^2p = +3rn^2p;$$

$$\text{The sum of which is } = -Kr - nmrp - ntrp + mtrp + rn^2p;$$

$$\text{Or } -nm+nt+mt \times r - m+t \times nrp + mt+nn \times rp,$$

$$\text{Or } -nm+nt+mt \times r - m+t - n \times nrp + mtrp,$$

$$\text{Or } -nm+nt+mt \times r + mt - m+t - n \times nrp;$$

Therefore the value of these three fractions will be,

$$\frac{-nm+nt+mt \times r - mt - m+t - n \times nrp}{nmt \times r - 1^2}, \text{ Or}$$

$$-nm+nt+mt - mt - m+t - n \times n \times p \times \frac{r}{nmt \times r - 1^2}.$$

But for the readier use of this hereafter, we may proceed thus; the sum of the three fractions whose denominators are $nmt \times r - 1^2$ is

$$\frac{-nm+nt+mt \times r - nmrp - ntrp + mtrp + rn^2p}{nmt \times r - 1^2},$$

$$\text{Or } \frac{-nm-nt-mt-nmrp-ntrp+mtrp+nrp}{nmt} \times 2 \text{ (by writ-}$$

ing 2 for $\frac{r}{r-1^2}$):

That

That is $2 \times \frac{\frac{1}{t} - \frac{1}{m} - \frac{1}{n} - \frac{p}{t} - \frac{p}{m} + \frac{p}{n} + \frac{np}{mt}}$

Or $2 \times \frac{\frac{1}{t} - \frac{1}{m} - \frac{1}{n} - \frac{1}{trn} - \frac{1}{mrn} + \frac{1}{nrn} + \frac{n}{mtrn}}$ by

restoring $\frac{1}{r^n}$ for p

The two fractions, whose common denominator is $mnt \times r - 1^3$ may (putting $n+m+t=1$) be added together as follows,

$$\begin{aligned} + 1 \times 1 - p \times \frac{1}{r+1} \cdot r &= \frac{1}{r+1} \cdot rs = \frac{1}{r+1} \cdot rp \times n+m+t, \\ + 1 \times 3np \times \frac{1}{r+1} \cdot r &= \frac{1}{r+1} \cdot rp \times 3n, \end{aligned}$$

The sum of which is $\frac{1}{r+1} \cdot rs - \frac{1}{r+1} \cdot rp \times n+t - 2n,$

Or $\frac{n+m+t-m+t-2n \times p \times \frac{1}{r+1} \cdot r}{mnt \times r - 1^3}$

Therefore the value of those two fractions will be

$$+ \frac{n+m+t-m+t-2n \times p \times \frac{1}{r+1} \cdot r}{mnt \times r - 1^3}$$

This may, also, be otherwise expressed, for the use of subsequent solutions, thus (putting $\frac{r+1 \times r}{r-1} = R$)

$$\frac{n+m+t-m+t-2n \times p}{mnt} \times R,$$

Or $R \times \frac{n+m+t-m+t-2np}{mnt}$

That

That is $R \times \frac{1}{mt} + \frac{1}{nt} + \frac{1}{nm} - \frac{p}{nt} - \frac{p}{nm} + \frac{2p}{mt}$;

Or $R \times \frac{1}{mt} + \frac{1}{nt} + \frac{1}{nm} - \frac{1}{mtr^n} - \frac{1}{nmr^n} + \frac{2}{mtr^n}$, by restoring $\frac{1}{r^n}$ for p .

And the value of the annuity will be,

$$\begin{aligned} \frac{1}{r-1} - \frac{1}{nm} + \frac{1}{nt} + \frac{1}{mt} - \frac{1}{m} + \frac{1}{n} + \frac{1}{t} - \frac{1}{n} \times \frac{1}{r} \times \frac{1}{p} \times \frac{r}{nmtr-1} \\ + \frac{n+m+t-m+t-2n \times p \times r + 1 \times r}{nmtr \times r-1^3} \\ - \frac{1-p \times r + 2 \times r - 3r}{nmtr \times r-1^4} \end{aligned}$$

The numerical operation deduced from this expression, contracted as below, will be the most expeditious; but the following (consisting of the two before-found expressions, with the substitution of

$$\frac{r+2 \times r-3r}{r-1^4} = S) \text{ will hereafter be wanted, viz.}$$

$$\begin{aligned} P + Q \times \frac{1}{t} - \frac{1}{m} - \frac{1}{n} - \frac{1}{tr^n} - \frac{1}{mtr^n} + \frac{1}{nr^n} + \frac{n}{mtr^n} \\ + R \times \frac{1}{mt} + \frac{1}{nt} + \frac{1}{nm} - \frac{1}{mtr^n} - \frac{1}{nmr^n} + \frac{2}{mtr^n} \\ = S \times \frac{1-p}{nmtr} \end{aligned}$$

Now,

Now, in order to shorten the first expression, substitute $K = am + nt + mt$, and $n + m + t = s$, as before.

Also $\frac{1}{r-1} = P$; $m + t = w$; and $\left(\frac{r}{amt \times r-1}\right)^2$

$\frac{PP_r}{amt} = L$: Then will, the annuity be equal to

$$\begin{aligned}
 & \frac{P - K - mt - w - n \times n \times \rho \times L}{+ s - w - 2n \times \rho \times r + 1 \times \rho \times L - \frac{1 - \rho \times r + 2^2}{-3 \times PP \times L}} \\
 \text{Or } P - L \times & \left\{ \begin{aligned} & \frac{K - mt - w - n \times n \times \rho}{- s - w - 2n \times \rho \times r + 1 \times P} \\ & + \frac{1 - \rho \times r + 2^2}{-3 \times PP} \end{aligned} \right. \\
 \text{Or } P - L \times & \left\{ \begin{aligned} & \frac{K - mt - w - n \times n \times \rho}{- P \times \left\{ \begin{aligned} & \frac{s - w - 2n \times \rho \times r + 1}{-1 - \rho \times r + 2^2} - 3 \times P \end{aligned} \right.} \end{aligned} \right.
 \end{aligned}$$

E X A M P L E.

What is the present value of an annuity of 1 *l.* to continue during the joint lives of three persons, whose ages are 43, 54, and 66, allowing compound interest at 4 *l. per Cent.*

Here

Here $n = (86-66) \div 20; m = (86-54) \div 32$; and $t = (86-43) \div 43$.

And $\pi=1.04$: $P=25$; $w=(32+43=) 75$; $p=0.456387$.

$$n=20; nm=640; mt=1376; r+1=2,04; u=75.$$
$$m \pm 32; \quad m = 860; \quad w - n \times r = 1100; \quad r + 2 = 3,04; \quad -2n = 40.$$

$t=43$; $m=1376$;

—	
—	
—	
—	Rems.
—	276; $\overline{r+2}^2 = 9,2416;$
—	35:

$s=95$; $K=2876$; $=3,000$; —

PP-625; —

Rems. 6, 2416.

Now.

Now $p = 0,456387$; $1-p = 0,543613$; $p = 0,456387$.

$$\begin{array}{r}
 \begin{array}{r}
 276 \\
 \hline
 125,962812 \\
 \hline
 K=2876 \\
 -125,962812 \\
 \hline
 \text{Rems. } 2750,037188 \\
 -1909,714825 \\
 \hline
 840,322363 \\
 \hline
 PP=625 \\
 r=1,04 \\
 \hline
 650 \\
 \hline
 \end{array}
 &
 \begin{array}{r}
 6,2416 \\
 \hline
 3,393015 \\
 \hline
 25 \\
 \hline
 84,825375 \\
 \hline
 161,213968 \\
 \hline
 84,825375 \\
 \hline
 76,388593 \\
 \hline
 25 \\
 \hline
 1909,714825 \\
 \hline
 \end{array}
 &
 \begin{array}{r}
 35 \\
 \hline
 15,973545 \\
 \hline
 95 \\
 \hline
 15,973545 \\
 \hline
 79,026455 \\
 \hline
 2,04 \\
 \hline
 161,213968 \\
 \hline
 32 \\
 \hline
 43 \\
 \hline
 1376 \\
 \hline
 20 \\
 \hline
 27520 \\
 \hline
 \end{array}
 \end{array}$$

$$\text{And } 840,322363 \times \frac{650}{27520} = 19,84775$$

Then $(25-19,84775=) 5,15225$, will be the present value of the annuity.

SCHOLIUM.

Since the above process is very tedious, it may be an agreeable service to shorten the same, by an approximation, similar to that in the scholium to quest. 64.

The series above given for the value of this annuity,

$$\text{viz. } \frac{n-1 \cdot m-1 \cdot t-1}{nmtr} + \frac{n-2 \cdot m-2 \cdot t-2}{nmtr^2} + \frac{n-3 \cdot m-3 \cdot t-3}{nmtr^3}$$

(n), may be considered as composed of the products of the corresponding terms of the three following series,

$$\frac{t-1}{t}$$

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$$\frac{1-1}{1} + \frac{1-2}{1} + \frac{1-3}{1} + \frac{1-4}{1} (n),$$

$$\frac{m-1}{m} + \frac{m-2}{m} + \frac{m-3}{m} + \frac{m-4}{m} (n),$$

$$\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} + \frac{n-4}{nr^4} (n);$$

The two first of which are arithmetical progressions, and the last (whose sum we suppose to be the known quantity N) may be taken as such, without a considerable error.

In the

$$\left\{ \begin{array}{l} \text{first} \\ \text{second} \\ \text{third} \end{array} \right\} \text{series, the greatest term is} \left\{ \begin{array}{l} \frac{1-1}{1} \\ \frac{m-1}{m} \\ \frac{n-1}{nr} \end{array} \right\} \text{the common difference} \left\{ \begin{array}{l} \frac{1}{1} \\ \frac{1}{m} \\ \frac{1}{nr} \end{array} \right\} \text{and the sum of the } n \text{ terms} \left\{ \begin{array}{l} n - \frac{n \cdot n + 1}{21} \\ n - \frac{n \cdot n + 1}{2m} \\ N. \end{array} \right.$$

Hence the expression, given in quest. 34, for the sum of n terms of such a series of products, viz.

$$\frac{SZW}{nn} + \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times a d \Delta + b d \Delta + c d \Delta - \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2}$$

$\times d \Delta$, may be applied to this case, by making

$$S = N \quad ; \quad a = \frac{n-1}{nr} \quad ; \quad \text{And } d = \frac{1}{nr} ;$$

$$Z = n - \frac{n \cdot n + 1}{2m} \quad ; \quad b = \frac{n-1}{m} \quad ; \quad \text{And } d = \frac{1}{m} ;$$

$$W = n - \frac{n \cdot n + 1}{2t} \quad ; \quad c = \frac{1-1}{1} \quad ; \quad \text{And } \Delta = \frac{1}{1} ;$$

Hence,

Hence

$$\frac{SZW}{sn} = (S \times \frac{Z}{n} \times \frac{W}{s}) = N \times \frac{n+1}{2m} \times \frac{n+1}{2t},$$

$$a\delta\Delta = \frac{n-1}{nr} \times \frac{1}{m} \times \frac{1}{t} = \frac{n-1}{nmr},$$

$$bd\Delta = \frac{n-1}{ns} \times \frac{1}{nr} \times \frac{1}{t} = \frac{n-1}{nmr},$$

$$cd\delta = \frac{t-1}{t} \times \frac{1}{nr} \times \frac{1}{m} = \frac{t-1}{nmr}, \text{ And}$$

$$d\delta\Delta = \frac{1}{nr} \times \frac{1}{m} \times \frac{1}{t} = \frac{1}{nmr}.$$

$$\frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times a\delta\Delta + bd\Delta + cd\delta \left\{ \begin{array}{l} = \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 3} \times \\ \frac{n-1 + n-1 + t-1}{nmr} \\ = \frac{n+1 \cdot n-1}{2 \cdot 6} \times \\ \frac{n+m+t-3}{mr} \end{array} \right.$$

$$\text{And } \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2} \times d\delta\Delta \left\{ \begin{array}{l} = \frac{n+1 \cdot n \cdot n-1}{2 \cdot 2 \cdot 2} \times \frac{1}{nmr} \\ = \frac{n+1 \times n-1}{2 \cdot 4} \times \frac{1}{mr} \end{array} \right.$$

$$\text{But } \frac{n+1 \cdot n-1}{2 \cdot 6} \times \frac{n+m+t-3}{mr} = \frac{n+1}{2m} \times \frac{n-1}{2r} \times \frac{n+m+t-3}{3t}$$

$$\text{And } \frac{n+1 \cdot n-1}{2 \cdot 4} \times \frac{1}{mr} = \frac{n+1}{2m} \times \frac{n-1}{2r} \times \frac{n-1}{2t},$$

$$\text{The difference of which will be } \left\{ \frac{n+1}{2m} \times \frac{n-1}{2r} \times \frac{n+m+t-3}{3t} - \frac{n-1}{2t} \right\}.$$

But

$$\text{But } \frac{n+m+t-3}{3t} - \frac{n-1}{2t} \left\{ \begin{aligned} &= \frac{2n+2m+2t-6-3n+3}{6t}, \\ &= \frac{2m+2t-n-3}{6t}; \end{aligned} \right.$$

Therefore the value of the annuity will be

$$N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} + \frac{n+1}{2m} \times \frac{n-1}{2t} \times \frac{2m+2t-n-3}{6t};$$

In which expression, if $\frac{n+1}{2m} = q$, it will become

$$N \times 1 - q \times 1 - \frac{n+1}{2t} + \frac{n-1}{6t} q \times \frac{m+t \times 2 - n + 3}{2t}.$$

The above, expressed in words at length, follows:

The Rule, for finding the present value of an annuity, to continue during the joint lives of three persons of given ages, allowing compound interest at a given rate, supposing the decrements of life to be equal, and the value of the oldest life to be given:

Let the number of years, that each of the persons want of 86, be called their complements of life, and let the sum of 1 l. and its interest for one year, be called the rate.

To the least complement add one, and divide the sum by the double of the next-greater complement, calling the quotient q ; let the same dividend be divided by twice the greatest complement, calling that quotient Q .

Subtract each of these quotients from unity; multiply the remainders together, and their product by the value of the oldest life, calling the product p .

Add the two greater complements together, and double their sum, from which subtract the least complement and three; multiply the remainder by the first found quotient q , and

and that product by the number which in table the last, stands on a line with the least complement less unity, and under the rate of interest.

Divide the last found product by twice the greatest complement, and let the last found quotient be added to the above product p , so shall the sum be the value of the annuity required.

EXAMPLE.

What is the present value of an annuity of 1*l.* to continue during the joint lives of three persons, whose ages are 43, 54, and 66; allowing compound interest at 4*l* per Cent. per Annum, the value of the oldest life being 7,333.

Here $(86 - 68 =)$ 20 the least complement,
 $(86 - 54 =)$ 32 the next greater complement,
 $(86 - 43 =)$ 43 the greatest complement,
 And $(1 + .04 =)$ 1,04 is the rate.

If $(20 + 1 =)$ 21 be divided by $(2 \times 32 =)$ 64, the quotient, q , will be ,328125; And if the same dividend, 21, be divided by $(43 \times 2 =)$ 86, the quotient, Q , will be ,244186.

The first quotient ($q =$) ,328125, being subtracted from unity, leaves ,671875, and the last quotient ($Q =$) ,244186, being subtracted from unity, leaves ,755814; which remainders being multiplied together, and their product being multiplied by 7,333, produces 3,7237 the product p .

The two greater complements (32, and 43) being added together make 75, the double of which sum is 150; from which, subtracting (the least complement more three =) 23 the remainder will be 127; which remainder

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der being multiplied by the first found quotient ($\frac{r}{r-1}$) 3,328125, and that product by 3,0449 (the number standing against (20—1=) 19, and under 4 *per Cent.* in table the last) will produce 126,8863; which being divided by (2×43=) 86, will quote 1,4754; to which adding the before found product 3,7237, their sum 5,1991, will be the value of the annuity required.

QUESTION LXX.

Supposing the decrements of life to be equal, required the present value of an annuity of 1 *l.* to continue during the joint lives of three persons, two of which are of the age of 54, and the other 66; allowing compound interest, at 4 *l. per Cent.*

This question may be answered from question 69, by $t=m$; for then

$$\begin{aligned}
 nm+nt+mt &= (nm+nm+mm) \frac{2n+m}{2n+m \times m}; \\
 m+t &= 2m; mt=mm; \text{ And } mt-m+t-n \times n = mm-2m-n \times n; \\
 \text{That is } mt-m+t-n \times n &= (mm-2mn+nn) \frac{m-n}{m-n^2}; \\
 annu &= am^2; n+m+t=n+2m; \text{ And } m+t-2n=m-n \times 2;
 \end{aligned}$$

Therefore the value of the annuity will be

$$\begin{aligned}
 \frac{1}{r-1} - \frac{2n+m \times m - m - n^2}{2n+m \times m - m - n^2} \times p \times \frac{r}{nm \times r - 1} \\
 + \frac{n+2m-2p \times m - n \times r + 1 \times r}{nm^2 \times r - 1} - \frac{1-p \times r + 2 \times 1 - 3r}{nm^2 \times r - 1} ;
 \end{aligned}$$

Which

Which expression (putting $\frac{r}{r-1} = P; m-n=d;$

And $\frac{r}{nm^2 \times r-1} = \frac{rPP}{nm^2} = L$) will become,

$$P-L \times \left\{ \begin{array}{l} 2n+m \times m - ddp, \\ -P \times \left\{ \begin{array}{l} n+2m-2pd \times r+1 \\ -1-p \times r+2^2-3 \times P. \end{array} \right. \end{array} \right.$$

SCHOLIUM.

Or, if the approximation (given in the scholium) be applied hereto, Then

$$\frac{n+1}{2r} = \frac{n+1}{2m} = q; m+1 \times 2 = n+3 = 4m-n+3; 8061 = 6m;$$

And the value of the annuity will be

$$N \times 1 - q + \frac{n-1}{6r} q \times \frac{4m-n+3}{2m}.$$

If the above question be answered by the first method; then $n=20; m=32; r=1,04; p=0,456387; \& P=25.$

$m-n=12=d; m+2n \times m=2304; \text{ and } ddp=65,7197$

Also $2n+m \times m - ddp = 2238, 280272.$

$n+2m=84; 2pd=10,953288; 84-10,953288=$

$73,046712; \text{ And } 73,046712 \times 2,04=149,015292:$

$1-p=0,543613; r+2=3,04 \text{ and } r+2^2-3=6,2416,$

Then $0,543613 \times 6,2416 \times 25=84,825275.$

Now $149,015292 - 84,825275 = 64,190017;$

$64,190017 \times 25=1604, 750425; rPP=650; \text{ and } nm^2=20480:$

$$\text{Then } \frac{2238, 280272 - 1604, 750425 \times \frac{650}{20480}}{20480} = 20,1072:$$

Lastly

Lastly $25 - 20,1071 = 4,8929$, will be the value required.

The latter method, expressed in words at length, may stand as follows :

The rule to find the present value of an annuity to continue during the joint lives of three persons, two of which are of equal ages, and the third of a greater age than either of them ; allowing compound interest at a given rate, supposing the decrements of life to be equal, and the value of the oldest life to be given.

Let the number of years, which each of the persons want of 86, be called their complements of life ; and let the sum of 11. and its interest for one year, be called the rate.

To the least complement add one, and divide the sum by twice the greater complement, calling the quotient q .

Subtract that quotient q from unity, multiply the remainder by itself, and that product by the given value of the oldest life, calling that last product p .

From four times the greater complement subtract the lesser complement, and the number 3 ; multiply the remainder by the number, which in the last table corresponds to the lesser complement less one, and rate. Also multiply that product by the above found quotient q ; then divide this last product by twice the greater complement.

To the last found quotient, add the before found product p , and the sum will be the value of the annuity required.

The same question answered by this rule.

Here $(86 - 56 =) 20$ the least complement.

$(86 - 54 =) 32$ the greatest complement.

And $(1 + ,04 =) 1,04$ the rate ; also $7,333 =$ value of the oldest life.

If $(20 + 1 =) 21$ be divided by $(2 \times 32 =) 64$, the quotient q will be $0,328125$.

The

The above quotient being taken from unity leaves 0,671875; which remainder multiplied by itself produces 0,451415, and that product multiplied by 7,333 will give 3,3102, which call p .

If from $(4 \times 32 =) 128$ be taken $(20 + 3 =) 23$, the remainder will be 105; Then $105 \times 3,0449 \times 0,328125 = 104,9062$; which product being divided by $(2 \times 32 =) 64 =$ will quote 1,6391.

Now $1,6391 + 3,3102 = 4,9493$, the value of the annuity required.

QUESTION LXXI.

Supposing the decrements of life to be equal; required the present value of an annuity of 1 l . to continue during the joint lives of three persons, two of which are of the age of 66, and the other 43; allowing compound interest at 4 l per Cent.

This question may be also answered from quest. 69, by making $m = n$; for then,

$$nm + nt + mt = nn + nt + nt = n + 2t \times n;$$

$$mt = nt; m + t - n = t; \text{ And } m + t - n \times n = tn;$$

$$\text{Th. } mt - m + t - n \times n = nt - nt = 0; nmt = n^2 t;$$

$$n + m + t = 2n + t; \text{ And } m + t - 2n = t - n;$$

Therefore the value of the annuity will be,

$$\frac{1}{r-1} - \frac{n + 2t \times n \times \frac{r}{n^2 t \times r - 1^2}}{1} + \frac{2n + t - t - n \times p \times r + 1 \times r}{n^2 t \times r - 1^3} - \frac{1 - p \times r + 2^2 \times r - 3r}{n^2 t \times r - 1^4}$$

Which expression (putting $\frac{r}{r-1} = P$

And $\frac{r}{nrf \times r-1} = \frac{PPr}{nrf} = L$) will become,

$$P-L \times \left\{ \begin{array}{l} \overline{n+2t} \times n \\ -P \times \left\{ \begin{array}{l} \overline{2n+r-1-n} \times p \times \overline{t+1} \\ -1-p \times r+2^2-3 \times P. \end{array} \right. \end{array} \right.$$

SCHOLIUM.

If we apply the method given in the scholium to quest. 69 hereto; Then-

$$\frac{n+1}{2n} = \frac{n+1}{2n}; \text{ And } \frac{\overline{n+1} \times 2 - \overline{n+3}}{6t} = \frac{2t+n-3}{6t};$$

Therefore the value of the annuity will be

$$N \times 1 - \frac{n+1}{2n} \times 1 - \frac{n+1}{2t} + \frac{n+1}{2n} \times \frac{n-1}{2r} \times \frac{2t+n-3}{6t};$$

$$\text{Now } 1 - \frac{n+1}{2n} = \frac{n-1}{2n}; \text{ And } 1 - \frac{n+1}{2t} = \frac{2t-n-1}{2t};$$

$$\text{Th. } N \times 1 - \frac{n+1}{2n} \times 1 - \frac{n+1}{2t} = N \times \frac{n-1}{2n} \times \frac{2t-n-1}{2t};$$

$$\text{That is } N \times \frac{2t-n-1}{4nt} \times \frac{n-1}{2n};$$

$$\text{Also } \frac{n+1}{2n} \times \frac{n-1}{2r} \times \frac{2t+n-3}{6t} = \frac{n+1}{6r} \times \frac{2t+n-3}{4nt} \times \frac{n-1}{2n};$$

Therefore the value of the annuity will be

$$N \times \frac{2t-n-1}{4nt} + \frac{n+1}{6r} \times \frac{2t+n-3}{4nt} \times \frac{n-1}{2n}.$$

The

The above question performed by the first method:
 where $x=20$; $t=43$; $r=1.04$; $p=0.456387$; $P=25$;
 $n+2t=106$; $106 \times 20=2120$; $2t+t=83$; $t-n=23$;
 $23 \times 0.456387=10.496901$, which taken from 83 leaves
 72.503099 ; And $72.503099 \times 2.04=147.906322$;
 $1-p=0.543613$; $r+2-3=6.2416$;
 And $0.543613 \times 6.2416 \times 25=84.8254$;
 Then $147.9063-84.8254 \times 25=1577.0225$;
 Also $2120-1577.0225 \times \frac{650}{17200}=20.5195$.
 Then $(25-20.5195)=4.4805$, is the value of the annuity
 required.

The latter method (expressed in words at length) may
 stand as follows.

*The Rule to find the present value of an annuity, to
 continue during the joint lives of three persons, two of
 which are of equal ages, and the third of a lesser age
 than either of them; allowing compound interest at a
 given rate, supposing the decrements of life to be equal;
 and the value of one of the elder lives to be given.*

*Let the number of years, which each of the persons
 want of 86, be called their complements of life, and let the
 sum of 1 l. and its interest for one year, be called the rate.*

*From twice the greater complement, take the lesser com-
 plement and one; multiply the remainder by the given
 value of the elder life, calling the product p.*

*To twice the greater complement, add the lesser, and
 from their sum subtract the number 3; multiply the remain-
 der by the number which, in the last table, corresponds to the
 lesser complement more one, and rate.*

*To this product, add the above found product p; multi-
 ply the sum by the lesser complement less one, and divide*

that product by four times the product of the three complements, so shall the quotient be the value of the annuity required.

Now if the same example be work'd by this last Rule,

Then $7,333$ is the value of one of the elder lives;

Also $(86-66=)$ 20 will be the least complement,

$(86-43=)$ 43 will be the greatest ditto,

And $(1+04=)$ $1,04$ will be the rate.

Now $(43 \times 2=)$ 86 ; and $(86-20-1=)$ 65 ; Then $(7,333 \times 65=)$ $476,65=p$.

Again $(2 \times 43=)$ 86 ; and $86+20=106$; Also $106-3=103$; Then if 103 be multiplied by $3,3654$ (the tabular number) it produces $346,635$.

If the above found product $476,645$, be added to the said product $346,635$; the sum will be $823,280$; which multiplied by $(20-1=)$ 19 produces $15642,320$; and that product being divided by $(4 \times 43 \times 20=)$ 3440 quotes $4,5471$, the value of the annuity required.

QUESTION LXXII.

Supposing the decrements of life to be equal; required the present value of an annuity of 1 l. to continue during the joint lives of three persons, each 66 years of age; allowing compound interest at 4 l. per Cent.

This question likewise may be answered from question 69, by making $n=m=t$; for then,

$$nm+nt+mt=3nn; \quad nt-m+t-n \times n=nn-nn=0; \\ n+nt+t=3n; \quad n+t-2n=0; \quad \text{and } nnt=n^3;$$

There-

Therefore the value of the annuity will be

$$\frac{1}{r-1} - 3n \times \frac{r}{n^3 \times r-1^2} + 3n \times \frac{r+1 \cdot r}{n^3 \times r-1^3} - \frac{1-p \times r + 2^2 \times r - 3r}{n^3 \times r-1^4},$$

$$\text{Or } \frac{1}{r-1} - \frac{3r}{n \times r-1} + \frac{3r \times r+1}{n \times r-1^3} - \frac{1-p \times r + 2^2 \times r - 3r}{n^3 \times r-1^4}.$$

Which expression (if $\frac{1}{r-1} = P$; $\frac{r}{r-1} = Q$, &c. as in corol. to question 19. will become

$$P - \frac{3}{n} Q + \frac{3}{n} R - \frac{1-p}{n^3} S.$$

If the method given in the scholium, be applied hereto; then,

$$\frac{n+1}{2n} = \frac{n+1}{2} = \frac{n+1}{2n} = q; \text{ And } \frac{n+1 \times 2 - n+3}{6} = \frac{n-1}{2n}.$$

And the value of the annuity will be,

$$\left(N \times \overline{1-q}^2 + \frac{n-1}{2r} q \times \frac{n-1}{2n} \right) N \times \overline{1-q}^2 + \frac{n-1^2 \times q}{4rn}.$$

$$\text{But } \overline{1-q} = \left(1 - \frac{n+1}{2n} = \frac{2n-n-1}{2n} \right) \frac{n-1}{2n};$$

Whence the value of the annuity will become,

$$N \times \frac{n-1^2}{4rn} + \frac{n-1^2}{4rn} \times \frac{n+1}{2n}, \text{ or } N \times \frac{n-1^2}{4rn} + \frac{n-1^2}{4rn} \times \frac{n+1}{2r};$$

$$\text{That is } N + \frac{n+1}{2r} \times \frac{n-1^2}{4rn}.$$

Now since $\overline{s-1}^2 = ns - 2n + 1$, And $s \times \overline{s-2} = ns - 2n$

Therefore $s \times \overline{s-2}$ may be wrote for $\overline{s-1}^2$ for the same reason as in quest. 65. and then the value of the annuity will become

$$N + \frac{n+1}{2r} \times \frac{s \times \overline{s-2}}{4ns}, \text{ Or } N + \frac{n+1}{2r} \times \frac{\overline{s-1}^2}{4n}$$

The above question answered by the first method.

Then $P=25$; $Q=269$; $R=33138$

$1-r \times S = 1378412$; And $n=20$:

$$\frac{650 \times 3}{20} = 97,5; \quad \frac{33150 \times 3}{20 \times 20} = 248,625; \text{ And}$$

$$\frac{1378412}{20 \times 20 \times 20} = 172,3015$$

$$\begin{array}{r} 25 \\ 248,625 \\ \hline \end{array} \quad \begin{array}{r} 97,5 \\ 172,3015 \\ \hline \end{array}$$

Then $273,625 - 269,8015 = 3,8235$, the value of the annuity required.

The latter method, expressed in words at length, may stand as follows.

The Rule to find the present value of an annuity, to continue during the joint lives of three persons of the same age; the value of a single life of that age being given; allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, be called the complement of life; and let the sum of 1. and it's interest for one year, be called the rate.

To the complement of life add one, and divide the sum by twice

twice the rate; or take three times the number which, in table the last, corresponds to the complement more one, and rate.

To the above quotient, or product, add the value of the single life, and multiply their sum by the complement less two; divide the product by four times the complement, and the quotient will be the value of the annuity required.

The above question answered by this rule.

Here $(86 - 66 =) 20$, will be the given complement of life.

7.333 , the value of the single life.

And $(1 - .04 =) .96$, the rate.

Then $(20 + 1 =) 21$ being divided by $(2 \times 1.04 =) 2.08$, will quote $(\text{or } 9.654)$ the tabular number being multiplied by 3 will produce 10.096 .

And $(10.096 + 7.333 =) 17.429$, being multiplied by $(20 - 2 =) 18$, produces 313.74 ; which being divided by $(4 \times 20 =) 80$, will give 3.9215 , for the value of the annuity required.

QUESTION LXXIII.

Supposing the decrements of life to be in a constant ratio; let there be three lives, the probabilities of the continuance of which for one year are respectively x , y , and z ; it is required to find the present value of an annuity, on their joint continuance, allowing compound interest to the purchaser.

SOLUTION.

By reasoning, as in quest. 66, the annuity may be expressed by the series

$$\frac{xyz}{r} + \frac{x^2y^2z^2}{r^2} + \frac{x^3y^3z^3}{r^3}, \text{ \&c.}$$

whose sum is $\frac{xyz}{r - xyz}$; found in the same manner as in that question.

L. 4.

QUES.

QUESTION LXXIV.

Having given the values of the annuities of three single lives (computed at a known rate of a interest) to find the value of an annuity, for their joint continuance, allowing compound interest to the purchaser?

SOLUTION.

Let the given values of the three annuities be N , M , and F ; and r the amount of 1*l.* in one year:

Then, supposing the decrements of life to be in a constant ratio, the probabilities of the continuance of those lives for one year will be respectively

$\frac{rN}{1+N}$, $\frac{rM}{1+M}$ and $\frac{rF}{1+F}$ by question 63, which expressions, being substituted in the stead of x , y , & z , in the last question, will give

$$\frac{rN}{1+N} \times \frac{rM}{1+M} \times \frac{rF}{1+F} \times \frac{1+N \times 1+M \times 1+F}{1+N \times 1+M \times 1+F \times r - r^3 NMF}$$

Or $\frac{rrNMF}{1+N \times 1+M \times 1+F - rrNMF}$, for the value of the annuity required.

What is the value of an annuity on three joint lives; which singly, computed at 4*l.* per Cent. are respectively worth 12,683; 10,478; and 7,333-

$$12,683 \times 11,478 \times 8,333 = 1,308,7266$$

$$12,683 \times 10,478 \times 7,333 \times 1,04 = 1,054,0198$$

$$254,7068) 1,054,0198$$

$$14,1382$$

SCHO

SCHOLIUM.

If this result be compared with the answer found per quest. 69; it will appear that this rule will give the value of the joint lives, considerably too little.

For the value of the 3 lives of the ages 43,	}	5,1523;
54, and 66 is		
By the above		4,1382,
Difference:		1,0141.

COROLL. I.

If the lives are of equal ages; then the value of the annuity, on their joint continuance, will be

$$\frac{rN^3}{1+N^3-rN^3}$$

COROLL. II.

Since the value of two equal joint lives is $\frac{rNN}{1+N^2-rNN}$; and the value of three equal joint lives $\frac{rrN^3}{1+N^3-rN^3}$; it will follow, that the value of m

equal joint lives will be $\frac{r^{m-1}N^m}{1+N^m-r^{m-1}N^m}$.

It may be thought, that (as in quest. 68. the value of two joint lives was computed, from the numbers given in the table of observations, deduced from the bills of mortality of London, so) we here should compute the va-

lue of three joint lives upon the same principle; but as we then found, that value to be between the results given by the two kinds of approximation, which have now been considered; we may reasonably expect, that the same will happen in this case; and consequently, that by taking a mean between them, we shall not be far from the truth.

QUESTION LXXV.

Supposing the decrements of life to be equal, the value of an annuity, on four equal joint lives, is required?

SOLUTION.

Let n denote the complement of each of those equal lives, and then it will appear, by reasoning as in the former questions, that the annuity will be worth

$$\frac{n-1^4}{n^4 \cdot 1} + \frac{n-2^4}{n^4 \cdot 2^2} + \frac{n-3^4}{n^4 \cdot 3^3} + \frac{n-4^4}{n^4 \cdot 4^4} (n), \text{ Or}$$

$$\begin{aligned} & \frac{n^4 - 4n^3 + 6n^2 - 4n + 1}{nnnnr} \\ & + \frac{n^4 - 8n^3 + 24n^2 - 32n + 16}{nnnnrr} \\ & + \frac{n^4 - 12n^3 + 54n^2 - 108n + 81}{nnnnrrr} (n); \end{aligned}$$

Which

Which may be divided into the five following series:

$$\begin{aligned} & \frac{n^4}{n^4} \times \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \frac{1}{r^5} (n) - \\ & \frac{4n^3}{n^4} \times \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \frac{5}{r^5} (n) + \\ & \frac{6n^2}{n^4} \times \frac{1}{r} + \frac{4}{r^2} + \frac{9}{r^3} + \frac{16}{r^4} + \frac{25}{r^5} (n) - \\ & \frac{4n}{n^4} \times \frac{1}{r} + \frac{8}{r^2} + \frac{27}{r^3} + \frac{64}{r^4} + \frac{125}{r^5} (n) + \\ & \frac{1}{n^4} \times \frac{1}{r} + \frac{16}{r^2} + \frac{81}{r^3} + \frac{256}{r^4} + \frac{625}{r^5} (n) \end{aligned}$$

Which series being summed by question 15, 16, 17, &c. will severally become

$$\frac{n^4}{n^4} \times \frac{1-p}{r-1} \text{ (where } p \text{ is the present worth of 1. due at the end of } n \text{ years.)}$$

$$\begin{aligned} & - \frac{4n^3}{n^4} \times \frac{1-p \times r}{r-1^2} - \frac{np}{r-1} \\ & + \frac{6n^2}{n^4} \times \frac{1-p \times rr+r}{r-1^3} - \frac{2np}{r-1^2} - \frac{np}{r-1} \\ & - \frac{4n}{n^4} \times \frac{1-p \times r^3+4r^2+r}{r-1^4} - \frac{3np \times rr+r}{r-1^3} - \frac{3np}{r-1^2} - \frac{n^3 p}{r-1} \\ & + \frac{1}{n^4} \times \left\{ \frac{1-p \times r^4+11r^3+11r^2+r}{r-1^5} - \frac{4np \times r^3+4r^2+r}{r-1^4} \right. \\ & \quad \left. - \frac{6np \times rr+r}{r-1^3} - \frac{4n^2 pr}{r-1^2} - \frac{n^4 p}{r-1} \right\} \end{aligned}$$

L 6

Which

Which sums being ranged according to their factors will become

$$\begin{aligned}
 & \frac{n^4 \times \overline{1-p}}{n^4 \times \overline{r-1}} \\
 & + \frac{4n^4 p}{n^4 \times \overline{r-1}} - \frac{4n^3 \times \overline{1-p} \times r}{n^4 \times \overline{r-1}^2} \\
 & - \frac{6n^4 p}{n^4 \times \overline{r-1}} - \frac{12n^3 p \times r}{n^4 \times \overline{r-1}^2} + \frac{6n^2 \times \overline{1-p} \times \overline{rr+r}}{n^4 \times \overline{r-1}^3} \\
 & + \frac{4n^4 p}{n^4 \times \overline{r-1}} + \frac{12n^3 p \times r}{n^4 \times \overline{r-1}^2} + \frac{12n^2 p \times \overline{rr+r}}{n^4 \times \overline{r-1}^3} \\
 & - \frac{n^4 p}{n^4 \times \overline{r-1}} - \frac{4n^3 p \times r}{n^4 \times \overline{r-1}^2} - \frac{6n^2 p \times \overline{rr+r}}{n^4 \times \overline{r-1}^3} \\
 & \left(\frac{4n \times \overline{1-p} \times r^3 + 4r^2 + r}{n^4 \times \overline{r-1}^4} \right) \\
 & \left(\frac{4np \times r^3 + 4r^2 + r}{n^4 \times \overline{r-1}^4} \quad \frac{1-p \times r^4 + 11r^3 + 11r^2 + r}{n^4 \times \overline{r-1}^5} \right)
 \end{aligned}$$

Now in these five fractions, whose denominators are $n^4 \times \overline{r-1}$, all the numerators are multiples of n^4 ; and consequently that quantity will vanish, and the denominator will be only $\overline{r-1}$. Also the numerators $\overline{1-p}$, $+4p$, $-6p$, $+4p$, and $-p$, being collected by addition, become unity; therefore the sum of these five fractions will be $\frac{1}{\overline{r-1}}$.

In those four fractions, whose denominators are $n^4 \times \overline{r-1}^2$, all the numerators are multiples of n^3 ; which being cancelled in both numerators and denominators,

tors, the denominators will become $n \times r-1^4$: And the numerators (omitting the common factor r) $-4, +4p, -12p, +12p$, and $-4p$, being added together become -4 ; therefore the sum of these four fractions will be

$$-\frac{4r}{n \times r-1^2}$$

In the three fractions, whose denominators are $n \times r-1^3$ (having cancelled nn in both numerators and denominators) the factors $6-6p, +12p$, and $-6p$, being added become $+6$, therefore the sum of those three fractions will be $+\frac{6 \times r-1r}{nn \times r-1^3}$.

In like manner the sum of the two fractions whose denominators are $n \times r-1^4$ will be

$$\frac{4 \times r^3 + 4r^2 + r}{n^3 \times r-1^4}$$

Therefore (putting $\frac{1}{r-1} = P; \frac{r}{r-1} = Q;$

$$\frac{r^2+r}{r-1^3} = R; \frac{r^3+4r^2+r}{r-1^4} = S; \frac{r^4+11r^3+11r^2+r}{r-1^5} = T)$$

The value of the annuity required will be

$$P - \frac{4}{n} Q + \frac{6}{nn} R - \frac{4}{n^3} S + \frac{1-p}{n^4} T.$$

C O R O L.

Retaining the above symbols; and putting N^{II} for the value of two equal joint lives whose complements are n ; N^{III} for three such joint lives; N^{IV} for four such, &c.

Since

$$\text{Given } N = P - \frac{1-p}{n} Q$$

by quest. 96.

$$N^{\text{II}} = P - \frac{2}{n} Q + \frac{1-p}{nn} R \quad 65.$$

$$N^{\text{III}} = P - \frac{3}{n} Q + \frac{3}{nn} R - \frac{1-p}{n^2} S \quad 72.$$

$$N^{\text{IV}} = P - \frac{4}{n} Q + \frac{6}{nn} R - \frac{4}{n^2} S + \frac{1-p}{n^3} T \quad 75.$$

$$\text{Therefore } \begin{cases} N^{\text{V}} = P - \frac{5}{n} Q + \frac{10}{nn} R - \frac{10}{n^2} S + \frac{5}{n^3} T - \frac{1-p}{n^4} V \\ N^{\text{VI}} = P - \frac{6}{n} Q + \frac{15}{nn} R - \frac{20}{n^2} S + \frac{15}{n^3} T - \frac{6}{n^4} V + \frac{1-p}{n^5} W. \end{cases}$$

And the value of an annuity, on m equal joint lives, will be

$$P - \frac{m}{n} Q + \frac{m \cdot m - 1}{n \cdot 2n} R - \frac{m \cdot m - 1 \cdot m - 2}{n \cdot 2n \cdot 3n} S \\ + \frac{m \cdot m - 1 \cdot m - 2 \cdot m - 3}{n \cdot 2n \cdot 3n \cdot 4n} T (\text{or}) + \frac{1-p}{n^m} Z.$$

In which expression, Z signifies the $m+1$ term in the series of factors $P, Q, R, S, \&c.$ whose values are found in questions 15, 16, 17, 18, $\&c.$ and the last term (viz. that which is multiplied by the factor Z) will be affirmative, when the term immediately preceding is negative; and negative, when the term immediately preceding is affirmative.

If the value of an annuity on 4, 5, 6, $\&c.$ unequal joint lives should be required; add their ages together, and take the 4th, 5th, 6th, $\&c.$ part of the sum (if

an

an integer) or the next lesser integer thereto, for a mean age; with which, find the value of the annuity as above; which will be sufficiently near the truth.

The reasons of proceeding no farther, in the computation of unequal lives, are, as follow:

First, Because questions of that kind seldom occur in practice, there being very few tenures, or leases on lives, wherein the number exceeds three.

Secondly, The computation derived from the principle, of the decrements of life being equal (which seems to be the only principle that can be applied thereto) would become excessive tedious; as would even the approximation, if it should (which is doubted) turn out to be shorter than the other: 'Tis true, if the values of the single lives be given, Mr. *De Moivre's* approximation may be used with tolerable exactness: And the manner of continuing thereof, to any number of joint lives is sufficiently evident. But this seems to differ more from the truth, the greater the number of lives is; indeed, when a computation is required to be made, from the Bills of mortality of *London*, it may be used with greater probability of exactness.

Thirdly, The greater the number of unequal lives are, the nearer will the result be found by the mean age; as might (if it needs a proof) be very easily made evident.

This method, of finding a mean age, may be useful, even in finding the values of two and three lives; when the ages given are not too far distant.

QUESTION LXXVI.

Supposing the decrements of life to be equal, it is required to find the value of an annuity, upon the longest of two lives, whose respective complements of life are m and n , m being the greater number?

SOLUTION.

Since (by the preceding) the probabilities of the continuance of those lives for one year are respectively

$\frac{n-1}{n}$ and $\frac{m-1}{m}$; Therefore the probabilities of their severally failing, will be $1 - \frac{n-1}{n}$, and $1 - \frac{m-1}{m}$ (by corol.

to quest. 26.) and consequently the probability of their both failing, in that time $1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m}$, which

probability, being subtracted from unity, will leave the probability of there being one of them (at least) existing at the year's end:

$$\text{But } 1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m} = 1 - \frac{n-1}{n} - \frac{m-1}{m} + \frac{n-1 \times m-1}{nm}$$

which, taken from unity, leaves $\frac{n-1}{n} + \frac{m-1}{m} -$

$\frac{n-1 \times m-1}{nm}$, the first payment of the annuity.

Again, since the probabilities of the given lives severally failing in the second year, are respectively

$1 - \frac{n-2}{n}$ and $1 - \frac{m-2}{m}$; therefore the probability of

their

their both failing will be $1 - \frac{n-2}{n} \times 1 - \frac{m-2}{m}$; which
 being taken from unity, will leave $\frac{n-2}{n} + \frac{m-2}{m}$ —
 $\frac{n-2 \times m-2}{nm}$ the second payment of the annuity.

By continuing the same kind of process, for the third,
 fourth, &c. years, and finding their present values, it will
 appear, that the required annuity may be expressed in
 the following manner.

$$\begin{aligned}
 & + \frac{n-1}{nr} + \frac{m-1}{mr} - \frac{n-1 \times m-1}{nmr} \\
 & + \frac{n-2}{nr^2} + \frac{m-2}{mr^2} - \frac{n-2 \times m-2}{nmr^2} \\
 & + \frac{n-3}{nr^3} + \frac{m-3}{mr^3} - \frac{n-3 \times m-3}{nmr^3} \\
 & + \frac{n-4}{nr^4} + \frac{m-4}{mr^4} - \frac{n-4 \times m-4}{nmr^4} \\
 & \text{&c.} \quad \text{&c.} \quad \text{&c.}
 \end{aligned}$$

But $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}$, &c. is the value of an
 annuity, on the single life whose complement is n (by
 quest. 56);

And $\frac{m-1}{mr} + \frac{m-2}{mr^2} + \frac{m-3}{mr^3}$, &c. is the value of an
 annuity, on the single life whose complement is m ;

Also $\frac{n-1 \times m-1}{nmr} + \frac{n-2 \times m-2}{nmr^2} + \frac{n-3 \times m-3}{nmr^3}$, &c.

is

The same performed by the last Rule.

Here $P = \left(\frac{1}{r-1} \right) 25$; $p = 0,285058$; $\frac{1}{r^m} = 0,185168$;

$n = 32$; $m = 43$; and $r+1 = 2,04$; $1-p = 0,714942$;

$\frac{1-p \times r+1 \times P}{n} = 1,139439$; And $rPP = 650$;

$$p + \frac{1}{r^m} = 0,470226;$$

Then the remainder $0,669213 \times \frac{650}{43} = 10,116$;

And $25 - 10,116 = 14,884$ the answer.

QUESTION LXXVII.

It is required to approximate to the value of the longest of two lives, supposing the decrements of life to be equal, and having the values of the single lives given.

SOLUTION.

Let the value of the life whose complement n is the lesser number, be denoted by N , and the value of the life whose complement is m , by M ;

Then the value of the joint lives will be

$$N - N \times \frac{n-1}{6r} \times \frac{r+1}{2m} \text{ by scholium to question 64.}$$

There-

Therefore from $M+N$ (the sum of the single lives)

Take
$$N - N \frac{n-1}{6r} \times \frac{n+1}{2m}$$

And the remainder
$$M + N \frac{n-1}{6r} \times \frac{n+1}{2m}$$

will be the value of the annuity, for the longest of the two lives.

EXAMPLE.

The values of the two single lives, of the ages 43 and 54; being 12,683, and 10,478; their complements 43, and 32; and the rate of interest 4 *per Cent.* what is the value of an annuity on the longest of those two lives?

In example 1. scholium to quest. 64. the value of the expression,

$$N \frac{n-1}{6r} \times \frac{n+1}{2m}$$
 was found to be 2,114;

To which add
$$M = 12,683;$$

The sum 14,797, will be the value of the annuity required, which differs from the true value above found, only by being 0,087 too little.

The Rule in words at length, will stand as follows:

The Rule to find the present value of an annuity, to continue during the life of the longest liver of two persons of given ages, allowing compound interest at a given rate, supposing

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supposing the decrements of life to be equal, and the values of both the single lives to be known.

Let the number of years, which each of the persons wants of 86, be called the complements of life; and let the sum of one pound, and its interest for one year, be called the rate.

From the lesser complement subtract one, and divide the remainder by six times the rate, or find this quotient in table the last.

From the value of the single life of the oldest person, subtract the above-found quotient, and multiply the remainder, by the lesser complement more one.

Divide the last found product, by twice the greater complement; and add the quotient to the value of the youngest life; so shall the sum be the value of the annuity required.

E X A M P L E.

What is the value of an annuity of one pound, to continue during the life of the longest liver, of two persons of the respective ages of 43, and 54, allowing compound interest at 4 per Cent. per Annum; the value of the life of 54 years being 10,478, and the value of the life of 43 years being 12,683:

Here $(86 - 43 =) 43$, is the greater complement;

And $(86 - 54 =) 32$, is the lesser complement;

Also $(1 + .04 =) 1.04$, is the rate:

Now if $(32 - 1 =) 31$, be divided by $(1.04 \times 6 =) 6.24$, the quotient will be 4, 970.

From 10,478, take 4,970, and there will remain 5,508; which multiplied by $(32 + 1 =) 33$ will produce 181,764.

If the last product (181,764) be divided by $(43 \times 2 =) 86$, the quotient will be 2,114.

Which

Which quotient, being added to 12,683, gives 14,797; for the value of the annuity required,

N.B. In this, and the following cases; the approximations, given by Mr. *De Moivre*, must (if used) be always applied to the first given rules (*viz.* those in which the values of the longest lives, and reversions, are directed to be found, by adding, or subtracting, the values of the single or joint lives) because the expressions of these approximations will not admit of being added, and subtracted, in the manner of the above.

QUESTION LXXVIII.

Supposing the decrements of life to be equal, it is required to find the value of an annuity, upon the longest of two equal lives, whose complement is n ?

S O L U T I O N.

This may be deduced from the solution of quest. 76; only by writing n for m , and p for $\frac{1}{r^n}$, as follows

$$P = \frac{1 - p \times r + r \times P}{n} - 2p \times \frac{rPP}{n}$$

Or (if deduced from the first expression there given)

$$P + 2 \times \frac{2p}{n} = R \times \frac{1-p}{rn}$$

E X.

EXAMPLE.

What is the value of the longest of two equal lives each of 48 years; allowing compound interest at 4 per Cent.

Here $n=38$; $r=1.04$; $p=0.225285$; $P=25$.

$$1-p \times r + 1 \times P = 39.510475, \text{ and } \frac{39.510475}{38} = 1.039749;$$

$$PPr = 650; \quad 2p = 0.450570;$$

$$1.039749 - 0.450570 = 0.589179;$$

$$\text{And } 0.589179 \times \frac{25 \times 25 \times 1.04}{38} = 10.07806;$$

Lastly $25 - 10.07806 = 14.92194$ the value of the annuity required.

QUESTION LXXIX.

'Tis required to approximate to the value of the longest of two equal lives, if the value of the single life be known?

SOLUTION.

From the value of the two equal lives $2N$;

Take the value of the equal joint lives (by quest. 65.) $\left\{ \frac{n-1 \times N}{2n} + \frac{n-1}{12r}; \right.$

And the value of the longest life will be $\left\{ 2N - \frac{n-1 \times N}{2n} - \frac{n-1}{12r}. \right.$

$$\text{Or } \frac{3n+1}{2n} N - \frac{n-1}{12r};$$

That

That is $\frac{37 \div 1 \times N}{2} - \frac{n-1}{6r} \times \frac{1}{2}$

Which, in words at length, follows.

The Rule, for finding the value of an annuity, upon the longest of two equal lives having the value of the single life given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given life wants of 86, be called the complement of life, and let the sum of 1 + 1, and its interest for one year, be called the rate.

To three times the complement add one; multiply the sum by the value of the single life; and divide that product by the complement.

From the complement subtract one, and divide the remainder by six times the rate; or find this quotient in table the last.

Then shall half the difference of those two quotients be the value of the annuity required.

If the example given in the last quest. be here propounded; then the value of the single life will be 11,748; (86—48=) 38 will be the complement of life; and 1,04 the rate.

Then (3×38+1=) 115, being multiplied by 11,748; produces 1351,02; which being divided by 38 gives 35,553 for the quotient.

Again (38—1=) 37, being divided by (6×1,04=) 624, will give 5 929 for the quotient.

And, half the difference of those quotients (viz. $\frac{35,553 - 5,929}{2} =$) 14,812 will be the value of the annuity required.

QUESTION LXXX.

Supposing the decrements of life to be equal; it is required to find the value of an annuity, upon the longest of three lives (that is to continue as long as either of them is in being) whose respective complements are t , m and n (t being $\leq m$, and $m \leq n$)?

SOLUTION.

As in quest. 76. $1 - \frac{n-1}{n}$, $1 - \frac{m-1}{m}$, and $1 - \frac{t-1}{t}$, will express the probabilities of the given live's, severally, failing in the first year, and therefore the probability of the failing of all of them the first year will be

$1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m} \times 1 - \frac{t-1}{t}$; which being subtracted from unity, will leave the probability of their being one of them (at least) existing at the year's end; which is the first payment of the annuity.

By continuing the process, expanding the expressions by multiplication, subtracting the products from unity, and finding the present values of the remainders, it will appear, that the value of the required annuity will be expressed by the following seven Series.

$$\frac{n-1}{n^2}$$

$$\begin{aligned} \frac{n-1}{nr} + \frac{m-1}{mr} + \frac{t-1}{tr} &= \frac{n-1 \times m-1}{nmr} - \frac{n-1 \times t-1}{ntr} \\ \frac{n-2}{nr^2} + \frac{m-2}{mr^2} + \frac{t-2}{tr^2} &= \frac{n-2 \times m-2}{nmr^2} - \frac{n-2 \times t-2}{ntr^2} \\ \frac{n-3}{nr^3} + \frac{m-3}{mr^3} + \frac{t-3}{tr^3} &= \frac{n-3 \times m-3}{nmr^3} - \frac{n-3 \times t-3}{ntr^3} \\ \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \end{aligned}$$

$$\begin{aligned} &\left(-\frac{n-1 \times t-1}{ntr} + \frac{n-1 \times m-1 \times t-1}{nmtr} \right) \\ &\left(-\frac{m-2 \times t-2}{mtr^2} + \frac{n-2 \times m-2 \times t-2}{nmtr^2} \right) \\ &\left(-\frac{m-3 \times t-3}{mtr^3} + \frac{n-3 \times m-3 \times t-3}{nmtr^3} \right) \\ &\text{&c.} \quad \text{&c.} \end{aligned}$$

Now $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}$, &c. is the value of that single life, whose complement is n ;

And $\frac{m-1}{mr} + \frac{m-2}{mr^2} + \frac{m-3}{mr^3}$, &c. is the value of that single life, whose complement is m ;

Also $\frac{t-1}{tr} + \frac{t-2}{tr^2} + \frac{t-3}{tr^3}$, is the value of that single life, whose complement is t ;

Again $\frac{n-1 \times m-1}{nmr} + \frac{n-2 \times m-2}{nmr^2} + \frac{n-3 \times m-3}{nmr^3}$, &c. is the value of those two joint lives, whose complements are n and m ;

$\frac{n-1 \times t-1}{ntr} + \frac{n-2 \times t-2}{ntr^2} + \frac{n-3 \times t-3}{ntr^3}$, &c. is the value of those two joint lives, whose complements are n and t ;

And $\frac{m-1 \times t-1}{mtr} + \frac{m-2 \times t-2}{mtr^2} + \frac{m-3 \times t-3}{mtr^3}$, &c. is the value of those two joint lives, whose complements are m and t ;

Lastly $\frac{n-1 \times m-1 \times t-1}{nmtr} + \frac{n-2 \times m-2 \times t-2}{nmtr^2}$, &c. is the value of the three joint lives.

Hence may be deduced the following rule, for finding the value of the longest of three lives, viz.

To the sum of the values of the three single lives, add the value of the three joint lives; and from their sum, subtract the three values of the joint lives, taken two and two, and the remainder will be the value of an annuity for the longest of the three lives.

EXAMPLE

If the lives are of the respective ages of 43, 54, and 66; Then

The value of a $\left\{ \begin{array}{l} 43 \\ 54 \\ 66 \end{array} \right\}$ is $\left\{ \begin{array}{l} 12,683 \\ 10,478 \\ 7,333 \end{array} \right\}$ by quest. 56.

Sum $30,494$
The value of the 3 joint lives is $\left\{ \begin{array}{l} 5,152 \text{ by quest. 69.} \end{array} \right\}$

Their sum $35,646$

Then

Then from that sum	35,646
Take { The value of 2 joint lives of 43 & 54 = 8,277	
{ And of 43 & 66 = 6,272	
{ And of 54 & 66 = 5,907	
Found per quest. 64	20,456
Remains the value of an annuity on the longest of the three lives	15,190

But as the above rule presupposes the solution of seven questions, four of which require long operations; it will be worth while to find a solution independent of any of them: In order to which it will be necessary to add and subtract the literal solutions, according to the form of the above numeral process.

Let therefore the values of the annuities on the single lives whose complements are n , m , and t , be denoted by N , M and F : The value of the joint lives whose complements are, n and m , by NM ; n and t , by MF ; and m and t , by NF : And the value of the annuity on the three joint lives by NMF .

Then $N+M+F-NM-NF-FM+NMF$ will be the value of the annuity required.

Now $N = P + Q \times \frac{1}{n} + \frac{1}{nr^n}$, by quest. 56.

$$M = P + Q \times \frac{1}{m} + \frac{1}{mr^m}$$

And $F = P + Q \times \frac{1}{t} + \frac{1}{tr^t}$

Whence $M+N+F$ will readily appear to

$$\text{be } 3P + Q \times \frac{1}{n} + \frac{1}{m} + \frac{1}{t} + \frac{1}{nr^n} + \frac{1}{mr^m} + \frac{1}{tr^t}$$

$$\text{Also } \overline{MNF} = N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} \times 1 - \left(+ \frac{n-1}{2r} \times \frac{n+1}{2m} \times \frac{m+t \times 2 - n+3}{6t} \right) \left. \begin{array}{l} \text{by schol. to} \\ \text{quest. 69.} \end{array} \right\}$$

The first term of which last expression, being expanded by multiplication, becomes—

$$N - \frac{n+1 \times N}{2m} - \frac{n+1 \times N}{2t} + \frac{n+1}{2m} \times \frac{n+1}{2t} N;$$

which being subtracted from the value of $\overline{NM} + \overline{NF}$

$$\text{will leave } N + \frac{n+1}{6r} \times \frac{n+1}{2m} + \frac{n-1}{6r} \times \frac{n+1}{2t} - \frac{n+1}{2m} \times \frac{n+1}{2t} N$$

$$\text{Or } N + \frac{1}{m} + \frac{1}{t} \times \frac{n-1 \times n+1}{12r} - \frac{n+1 \times n+1}{4mt} N;$$

Now, for the second term of the value of

$$\overline{NMF} \left(\text{viz. } \frac{n-1}{2r} \times \frac{n+1}{2m} \times \frac{m+t \times 2 - n+3}{6t} \right)$$

$$\text{may be wrote } \frac{n-1 \times n+1}{12r} \times \frac{m+t \times 2 - n+3}{2mt}; \text{ which being}$$

subtracted from the second term of the above remainder

$$\left(\text{viz. } \frac{1}{m} + \frac{1}{t} \times \frac{n-1 \times n+1}{12r} \right) \text{ there will remain}$$

$$\frac{1}{m} + \frac{1}{t} \times \frac{m+t \times 2 - n+3}{2mt} \times \frac{n-1 \times n+1}{12r},$$

$$\text{Or } \frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r}.$$

And then, the value of $\overline{NM} + \overline{NF} - \overline{NMF}$ will be

$$N + \frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r} - \frac{n+1 \times n+1}{4mt} N;$$

Therefore

Therefore the value of $NM + NF + MF - NMF$ will be

$$N + \frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r} + \frac{n+1^2 \times N}{4mt}$$

$$\left(+ M - \frac{n+1 \times M}{2t} + \frac{n-1 \times m+1}{12rt} \right),$$

$$\text{Or } N + M - \frac{n+1^2 \times N}{4mt} - \frac{n+1 \times M}{2t} + \frac{n-1 \times m+1}{12rt}$$

$$\left(+ \frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r} \right);$$

Which being subtracted from $N + M + F$ will leave,

$$F + \frac{n+1^2 \times N}{4mt} + \frac{n+1 \times M}{2t} - \frac{n-1 \times n+1}{12rt}$$

$$\left(- \frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r} \right), \text{ the value of the annuity.}$$

Now, since the above approximation to the value of the annuity is found by subtracting of three approximations (*viz.* those to the 3 values of two joint lives) each of which exceeds the truth; and by the adding only of one approximation (*viz.* that to the value of three joint lives) which is also greater than the truth: it will follow that the value above found will be less than just:

And since $\left(\frac{n+3}{2mt} \times \frac{n-1 \times n+1}{12r} \right) - \frac{n-1}{6r} \times \frac{n+3 \times n+1}{4mt}$ is a quantity, to be subtracted, in the above approximation; therefore if that quantity be lessened, the approximation will be encreased,

But that quantity will not be lessened (and consequently the approximation not encreased) by $\frac{1}{12r}$, if $\frac{n-1}{6r} \times \frac{n+1}{4mt}$

be wrote in its stead : For $\frac{n-1 \times n+1}{6r} \times \frac{n+1}{24r}$ may be wrote for the said expression ; where $\left(\frac{n-1 \times n+1}{6r} = \right)$

$\frac{n-1}{6r}$ must be less than unity, because both r and m are

greater than n ; and by writing $\frac{n+1}{24r}$ for $\frac{n+1}{24r}$, we lessen

that part of the expression by $\left(\frac{2}{24r} = \right) \frac{1}{12r}$; which, because r is greater than unity, is less than $\frac{1}{12}$: And consequently the whole will not be lessened by $\frac{1}{12}$ of unity.

Putting therefore $\frac{n-1}{6r} \times \frac{n+1}{4mt}$, for $\frac{n+1}{24r} \times \frac{n-1 \times n+1}{12r}$, the approximation to the value of the annuity will become

$$F + \frac{n+1}{4mt} N + \frac{m+1 \times M}{2t} - \frac{n-1 \times m+1}{12rt} - \frac{n-1}{6r} \times \frac{n+1}{4mt}$$

$$\text{Or } F + \frac{n+1}{4mt} N - \frac{n+1}{4mt} \times \frac{n-1}{6r} + M \times \frac{m+1}{2t} - \frac{n-1}{6r} \times \frac{n+1}{4mt}$$

$$\text{That is } F + N \times \frac{n-1}{6r} \times \frac{n+1}{4mt} + M \times \frac{m+1}{6r} \times \frac{n+1}{2t}$$

$$\text{Or } F + \frac{1}{2t} \times N - \frac{n-1}{6r} \times \frac{n+1}{4mt} + M \times \frac{m+1}{6r} \times \frac{n+1}{2t}$$

Which

Which, expressed in words at length, follows:

The rule for finding the value of an annuity, on the longest of three unequal lives, of given ages; having the values of those single lives given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life; and let r , and its interest for one year, be called the Rate.

From the least complement subtract one, and divide the remainder by six times the rate; or find this quotient in table the last; subtract this quotient from the value of the eldest life, reserving the remainder.

To the lesser complement add one, multiply the sum by itself; and that product by the remainder above reserved; dividing this last product by twice the middle complement, reserving the quotient.

From the middle complement subtract one, and divide the remainder by six times the rate; or find the quotient by table the last; subtract the quotient from the value of the middle life, and multiply the remainder by the middle complement more one.

To the last found product, add the quotient above reserved, and divide their sum by twice the greatest complement; to which quotient, add the value of the youngest life, and the sum will be the value of the annuity required.

The operation of the example, given in quest. 80, by this rule, may stand as follows.

Here the value of the youngest life will be	12,683
of the middle	10,478
of the eldest	7,333

And $(86 - 43 =) 43$ will be the greatest complement,
 $(86 - 54 =) 32$ middle
 $(86 - 66 =) 20$ least

Also $(1 + .04 =) 1.04$, the rate.

Now, if $(20 - 1 =) 19$ be divided by $6 \times 1.04 = 6.24$, the quotient will be 3,045; which, subtracted from 7,333, leaves 4,288, for the remainder to be reserved.

And, if $(32 - 1 =) 31$ be multiplied by 21, it produces 651; and 651 multiplied by the above reserved remainder 4,288, produces 2,791,008; which, being divided by $(2 \times 32 =) 64$ will give 29,547, for the quotient to be reserved.

Also, if $(32 - 1 =) 31$ be divided by 6,24, the quotient will be 4,968; which, subtracted from 10,478, leaves 5,510; and this multiplied by $(32 - 1 =) 31$, produces 181,830.

Again, the last product 181,830, being added to 29,547, the reserved quotient, makes 211,377; which, divided by $(43 \times 2 =) 86$, quotes 2,458; and the sum of 12,683 & 2,458 (*viz.* 15,141) will be the value of the annuity required.

QUEST.

QUESTION LXXXII.

Supposing the decrements of life to be equal, it is required to find the value of an annuity, upon the longest of three lives; two of which are of equal ages, and younger than the third.

The solution of this question may be deduced from quest. 80, by writing m for s ; and $\frac{1}{r^m}$, for $\frac{1}{r^s}$; whence the value of the annuity required will be

$$P + \frac{rP^2}{m} \times \left\{ \frac{\frac{m}{m} \times p + \frac{2}{r^m} + 2p + \frac{1}{r^m} + \frac{r+1 \times P}{m}}{1 - 2pr + 2 - 3 \times P^2} \right\}$$

For example; Let the given equal ages be 54, and the elder life 66; also let compound interest be allowed at 4 per Cent.

Then $m=32$; $r=20$; $\frac{1}{r^m}=1,04$; $p=0,456387$; $P=25$,
 $1-p=0,543613$; $r+2-3=6,2416$; $nm=640$;
 $\frac{1}{r^m}=0,285058$; $\frac{m}{m}=0,625$; $0,625 \times 0,456387 =$
 $0,285841$;
 $2 \times 0,456387 = 0,912774$; and $0,912774 + 0,285058 =$
 $1,197832$;

Again, $1,197832 \times \frac{2,04 \times 25}{32} = 1,90985$; $\frac{PPr}{m} = 20,3123$
 $\frac{0,543613 \times 6,2416 \times 25 \times 25}{20 \times 32} = 3,313491$;

And

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And $0,285241 + 0,570116 + 1,90905 = 2,76441$;

Also $2,76441 - 3,31349 = -0,54908$;

$-0,54908 \times 20,3125 = -11,1532$;

Lastly, $25 - 11,1532 = 13,8468$, is the value of the annuity required.

SCHOLIUM.

If the approximation, given in the last question, be applied to the value of the annuity, by writing m for t and M for F , it will become

$$M + \frac{1}{2m} \times N - \frac{1}{6} \times \frac{1}{2m} \times \frac{1}{2m} + M - \frac{m-1}{6m} \times \frac{1}{2m} + 1$$
. The expression of which, in words at length, will differ from the former, only, in writing, greater complement, instead of greatest and middle complement; and younger life, for middle and youngest life.

QUESTION LXXXIII.

Supposing the decrements of life to be equal, it is required to find the value of an annuity upon the longest of three lives, two of which are of equal ages, and elder than the third.

The solution of this question may also be deduced from quest 80, by writing a for m , and p for $\frac{1}{m}$; whence the value of the annuity will be,

$$P + \frac{rP^2}{r} \times \left\{ \frac{\frac{1}{r^2} + 2 + \frac{r+1 \times 3P}{n}}{1 - \frac{r \times r}{r^2} + 2^2 - 3 \times P^2} \right.$$

$$\text{Or } P + \frac{rP^2}{r} \left\{ \frac{\frac{1}{r^2} + 2 + \frac{r+1 \times 3P}{n}}{1 - \frac{r \times r}{r^2} + 2^2 - 3 \times P^2} \right. ; \text{ That is}$$

$$P + \frac{rP^2}{r} \times \left\{ \frac{\frac{1}{r^2} + 2 + \frac{r+1 \times 3P}{n}}{1 - \frac{r \times r}{r^2} + 2^2 - 3 \times P^2} \right.$$

For example, let the given equal ages be 66; and the younger 43; and let compound interest be allowed at 4 per Cent.

Here $t=43$; $p=0.456387$; $r=0.543613$; $P=25$;
 $n=20$; $\frac{1}{r^2}=0.185168$; $\frac{rPP}{t}=\frac{650}{43}=15.11628$; $r+1=$
 2.04 ; $2.04 \times 3=6.12$; $6.12 \times \frac{25}{20}=7.65$; $7.65+2 \times$
 $0.456387=4.404134$; $r+2^2-3=6.2416$;

And $\frac{0.543613 \times 6.2416 \times 25 \times 25}{20 \times 20}=5.301586$;

New $0.185168+4.404134-5.301586=0.712284$;

And $15.11628 \times 0.712284=10.76208$;

Lastly

Lastly, $25 - 10,767,08 = 14,232,92$, the value of the annuity required.

QUESTION LXXXIV.

Supposing the decrements of life to be equal, it is required to approximate to the value of an annuity upon the longest of three lives, two of which are equal, and elder than the third.

SOLUTION.

This may be readily solved, by writing N for M , and n for m , in the result of quest. 81; whence the value of the annuity will be

$$F + \frac{1}{2t} \times N - \frac{n-1}{6r} \times \frac{n+1}{2n} + N - \frac{n-1}{6r} \times n + 1$$

$$\text{Or } F + \frac{1}{2t} \times N - \frac{n-1}{6r} \times \frac{n+1}{2n} + n + 1$$

$$\text{Now } \frac{n+1}{2n} + n + 1 = \left(\frac{n+1}{2n} + 1 \right) \times n + 1 = \frac{3n+1 \times n+1}{2n}$$

Th. $F + \frac{3n+1 \times n+1}{4nt} \times N - \frac{n-1}{6r}$ will be the value of the annuity.

Which may be expressed in words, at length, as follows.

The rule for finding the value of an annuity, on the longest of three lives, two of which are equal and elder than the third, having the values of the single lives given, allowing interest

interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life, and let one pound, and its interest for one year, be called the rate.

From the lesser complement subtract one, and divide the remainder by six times the rate, or find this quotient by table the last; subtract this quotient from the value of one of the elder lives, reserving the remainder.

To three times the lesser complement add one, and multiply the sum by the same complement more one; also multiply this product by the remainder above reserved, and divide the quotient by four times the product of the two complements.

To the last found quotient, add the value of the younger life, and the sum will be the value of the annuity required.

The operation of the example, given in quest. 83, may stand as follows.

Here the value of the younger life will be 12,683
 elder 7,333
 And $(86-43=)$ 43 will be the greater complement,
 $(86-66=)$ 20 lesser
 Also $(1+.04=)$ 1,04 the rate.
 Now if $(20-1=)$ 19, be divided by $(6 \times 1,04=)$ 6,24,
 the quotient will be 3,045; which, subtracted from 7,333,
 leaves 4,288, for the remainder to be reserved.

And if $(3 \times 20+1=)$ 61 be multiplied by $(20+1=)$ 21, the product will be 1281; which, being also multiplied by 4,288, will produce 5492,928.

Now

Now $(4 \times 43 \times 20 =) 3440$ is the product of four times the two complements; by which, if 5492,928 (the number above found) be divided, the quotient will be 1,597; to which adding 12,681 (the value of the younger life) the sum, 14,280, will be the value of the annuity required.

QUESTION LXXXV.

Supposing the decrements of life to be equal, it is required to find the value of an annuity, upon the longest of three equal lives.

The solution of this may be also reduced from quest. 80, by writing a for x and a , and p for $\frac{1}{r^n}$ and $\frac{1}{r^2}$; and then the value of the annuity will be

$$\frac{1}{r-1} + \frac{3p}{n} \times \frac{r}{r-1} + \frac{3p}{nn} \times \frac{r+1 \times r}{r-1} - \frac{1-p}{r^2} \times \frac{r+2 \times r-3r}{r-1}, \text{ Or}$$

$$P + \frac{3p}{n} Q + \frac{3p}{nn} R - \frac{1-p}{n^2} S, \text{ that is}$$

$$P + \frac{rP^2}{n} \times \left\{ \frac{3p + 3p \times \frac{r+1 \times P}{n}}{\frac{1-p \times r + 2^2 - 3 \times PP}{nn}} \right\}; \text{ that is}$$

$$P + \frac{rP^2}{n} \times \left\{ \frac{\frac{r+1 \times P}{n} \times 3P}{\frac{1-p \times r + 2^2 - 3 \times PP}{nn}} \right\}$$

For example; let the three persons be each 54 years old, and compound interest be allowed at 4% *per Cent.*

$$\begin{aligned} \text{Then } 32 &= A; 0,285058 = P; 1,04 = r; 0,714942 = \\ 1 - 21 &= Q; 25 = P; (r+1) = 2,04 \times 25 = 51,00; \text{ and } \\ \frac{51}{32} &= 1,59375; \text{ also } 32 = 0,855174; 0,855174 \times 1,59375 \\ &= 2,21807; r+2 = 3 = 6,2416; 6,2416 \times 0,714942 = \\ 4,4624; \text{ and } 4,4624 \times \frac{25 \times 25}{32 \times 32} &= 2,723649; \end{aligned}$$

$$\text{Now } 2,723649 - 2,21807 \times \frac{25 \times 25 \times 1,04}{32} = 10,2688;$$

Whence $25 - 10,2688 = 14,7312$, the value of the annuity required.

SCHOLIUM I.

If we make use of the approximation in quest. 84, by writing a for t and N for E , the value of the annuity will be

$$N + \frac{3n+1 \times n+1}{4nn} \times N - \frac{n-1}{6r} \text{ which may be expressed in words in the same manner as that remembering that the lives and the complements are equal.}$$

SCHOLIUM II.

If it were required to find the value of an annuity, to continue as long as any two (out of three persons of given ages) shall be alive; the same might be investigated as follows.

If

If their complements of life be represented by l , m , and n , as before.

Then the probability of the continuance, for one year, of the lives of

$\left\{ \begin{array}{l} \text{all three will be} \\ \text{the two eldest} \\ \text{the eldest and youngest} \\ \text{the two youngest} \end{array} \right.$	all three will be	$\frac{l-1}{l} \times \frac{m-1}{m} \times \frac{n-1}{n}$
	the two eldest	$\frac{m-1}{m} \times \frac{n-1}{n} \times 1 - \frac{l-1}{l}$
	the eldest and youngest	$\frac{l-1}{l} \times \frac{n-1}{n} \times 1 - \frac{m-1}{m}$
	the two youngest	$\frac{l-1}{l} \times \frac{m-1}{m} \times 1 - \frac{n-1}{n}$

Which kind of process being continued, and the present worths being found, it will appear, that an annuity during the joint continuance of two lives out of three, will consist of the three joint lives of the given persons taken two and two, less twice the joint lives of all the three.

The actual addition and subtraction of the respective values of the unequal lives (as the result will not be often wanted) is left as an exercise to the reader; but is inserted for equal lives, because that operation will take up but little room; thus,

From three times the value of two equal joint lives, $\left\{ 3 \times \frac{2 \times 2}{n} - 2 \times \frac{1 \times 1}{n} \right\} R$,

Take

Take twice the val. of } $2P = \frac{2 \times 3}{n} Q + \frac{2 \times 3}{n^2} R - \frac{2 \times 1 - 1}{n^3} S$;
three equal joint-lives

Remains the value of } $P = \frac{3 \times 1 + 1}{n^2} R + \frac{2 \times 1 - 1}{n^3} S$.
an annuity to continue
as long as 2 of 3 lives.

QUESTION LXXXVI.

Supposing the decrements of life to be equal, it is requir'd to find the value of an annuity, upon the longest of four equal lives.

SOLUTION.

Let n be the complement of life, then will $1 - \frac{n-1}{n}$

express the probability of either of them failing in the first year; and therefore the probability of the failing of all of them in that year will be

$$\left(1 - \frac{n-1}{n}\right)^4 = 1 - 4 \times \frac{n-1}{n} + 6 \times \frac{n-1}{n^2} - 4 \times \frac{n-1}{n^3} + \frac{n-1}{n^4}$$

which, being subtracted from unity, will leave,

$$4 \times \frac{n-1}{n} - 6 \times \frac{n-1}{n^2} + 4 \times \frac{n-1}{n^3} - \frac{n-1}{n^4}, \text{ for the probability, that one (at least) of them will be alive at the year's end; therefore the first payment will be worth}$$

$4 \times \frac{n-1}{nr} - 6 \times \frac{n-1}{n^2 r} + 4 \times \frac{n-1}{n^3 r} - \frac{n-1}{n^4 r}$: And by continuing the process, the second; third; &c. years payments will be worth

$$4 \times \frac{n-2}{nr^2} - 6 \times \frac{n-2}{n^2 r^2} + 4 \times \frac{n-2}{n^3 r^2} - \frac{n-2}{n^4 r^2}$$

$$4 \times \frac{n-3}{nr^3} - 6 \times \frac{n-3}{n^2 r^3} + 4 \times \frac{n-3}{n^3 r^3} - \frac{n-3}{n^4 r^3}$$

8cc

Now $\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3}$ (n) is the value of the single life,
whose complement is n ;

And $\frac{n-1}{n^2 r} + \frac{n-2}{n^2 r^2} + \frac{n-3}{n^2 r^3}$ (n) is the value of two equal joint
lives, whose complements are n ;

$\frac{n-1}{n^3 r} + \frac{n-2}{n^3 r^2} + \frac{n-3}{n^3 r^3}$ (n) is the value of three equal joint
lives, whose complements are n ;

And $\frac{n-1}{n^4 r} + \frac{n-2}{n^4 r^2} + \frac{n-3}{n^4 r^3}$ (n) is the value of four equal
joint lives, whose complements are n .

Therefore, if, from four times the value of the single life,
six times the value of two equal joint lives be taken; and
then, to the remainder, four times the value of three equal
joint lives be added; and lastly, from the sum, the value
of the four equal joint lives be taken; then shall the last re-
mainder be the value of the longest of the four equal lives.

Now, if we assume the same symbols as in question 20.
then,

$$4N = 4P - \frac{4-4p}{n} Q;$$

$$-6N^{ii} = -6P + \frac{12}{n} Q - \frac{6-6p}{nn} R;$$

$$+4N^{iii} = 4P - \frac{12}{n} Q + \frac{12}{nn} R - \frac{4-4p}{n^3} S;$$

$$-N^{iv} = -P + \frac{4}{n} Q - \frac{6}{nn} R + \frac{4}{n^3} S - \frac{1-p}{n^4} T;$$

Therefore, the value of the longest of the four equal lives will be

$$P + \frac{4p}{n} Q + \frac{6p}{nn} R + \frac{4p}{n^3} S - \frac{1-p}{n^4} T$$

C O R O L.

Retaining the above symbols, and writing $L.N^{ii}$ for the value of the longest of two equal lives, whose complements are n ; $L.N^{iii}$ for the value of the longest of three such lives; $L.N^{iv}$, for the value of four such, &c.

$$\text{Since } N = P - \frac{1-p}{n} Q; \quad \text{by quest. 56}$$

$$L.N^{ii} = P + \frac{2p}{n} Q - \frac{1-p}{nn} R; \quad 78$$

$$L.N^{iii} = P + \frac{3p}{n} Q + \frac{3p}{nn} R - \frac{1-p}{n^3} S; \quad 85$$

$$L.N^{iv} = P + \frac{4p}{n} Q + \frac{6p}{nn} R + \frac{4p}{n^3} S - \frac{1-p}{n^4} T; \quad 186$$

Therefore

$$\begin{cases} L.N = P + \frac{5p}{n}Q + \frac{10p}{n^2}R + \frac{10p}{n^3}S + \frac{5p}{n^4}T - \frac{1-p}{n^5}V, \\ L.N^2 = P + \frac{6p}{n}Q + \frac{15p}{n^2}R + \frac{20p}{n^3}S + \frac{15p}{n^4}T + \frac{6p}{n^5}V - \frac{1-p}{n^6}W. \end{cases}$$

And the value of an annuity, on the length of m equal lives will be

$$P + \frac{mp}{n}Q + \frac{mp \cdot m-1}{n \cdot 2n}R + \frac{mp \cdot m-1 \cdot m-2}{n \cdot 2n \cdot 3n}S \\ + \frac{mp \cdot m-1 \cdot m-2 \cdot m-3}{n \cdot 2n \cdot 3n \cdot 4n}T - \frac{1-p}{n^m}Z.$$

Where Z signifies the $m+1$ term of the scale of factors, P, Q, R, S , &c. whose values are found in quest. 15, 16, 17, &c.

QUESTION

QUESTION LXXXVII.

Supposing the decrements of life to be equal ; it is required to find the value of an annuity, to continue as long as any three (out of four equal lives) shall be in being.

SOLUTION.

If n be the complement of life, then will $\frac{n-1}{n^4}$ be the probability of the continuance of all the four lives for one year ; $\left(\frac{n-1}{n^3} \times 1 - \frac{n-1}{n} \right) \frac{n-1}{n^3} - \frac{n-1}{n^4}$ will be the probability of the failing of some one (particularly named) and of the continuance of the other three for one year ; and because there are four persons, each of which may be so. particularly named, therefore that probability must be taken four times.

Th. $\left(\frac{n-1}{n^4} + \frac{n-1}{n^3} \frac{n-1}{n^4} \times 4 = \right) \frac{n-1}{n^3} \times 4 - \frac{n-1}{n^4} \times 3$
will be the value required.

That is, if from four times the value of three equal joint lives, three times the value of the four equal joint lives be taken, the remainder will be the value of an annuity to continue as long as any three of them shall be alive.

Now four times the value of $\left\{ \begin{array}{l} 4P - \frac{4X^3}{n} Q + \frac{4X^3}{nn} R - \frac{4X^{1-p}}{n^3} S; \end{array} \right.$

And three times the value of $\left\{ \begin{array}{l} 3P - \frac{3X^4}{n} Q + \frac{3X^6}{nn} R - \frac{3X^4}{n^3} S + \frac{3X^{1-p}}{n^4} T; \end{array} \right.$

Remains the value of three $\left\{ \begin{array}{l} P - \frac{6}{nn} R + \frac{2X^4+4p}{n^3} S - \frac{3X^{1-p}}{n^4} T. \end{array} \right.$

COROL.

From the two last processes we may find the value of an annuity, to continue as long as $\overline{m-1}$ out of m equal lives (whose complements are n) shall be in being; thus, put N^m for the value of m equal joint lives; and N^{m-1} for the value of $\overline{m-1}$ equal joint lives.

From

$$\text{From } m \times N^{m-1} = mP - \frac{m \times m - 1}{n} Q + \frac{m \cdot m - 1 \cdot m - 2}{n \cdot 2n} R$$

$$\left(\text{etc. } + \frac{m \times 1 - p}{n^{m-1}} r, \right.$$

$$\text{Take } m-1 \times N^m = m-1 \times P - \frac{m-1 \times m}{n} Q + \frac{m-1 \cdot m \cdot m - 1}{n \cdot 2n} R$$

$$\left(R \text{ etc. } + \frac{m-1 \times m}{n^{m-1}} r + \frac{m-1 \times 1 - p}{n^m} Z, \right.$$

Remains the value of the annuity required.

$$= P - \frac{m \cdot m - 1}{n \cdot 2n} R + \frac{2m \cdot m - 1 \cdot m - 2}{n \cdot 2n \cdot 3n} S - \frac{3m \cdot m - 1 \cdot m - 2 \cdot m - 3}{n \cdot 2n \cdot 3n \cdot 4n} T$$

$$\left(\text{etc. } + \frac{m-2 + p \times m}{n^{m-1}} r + \frac{m-1 \times 1 - p}{n^m} Z. \right.$$

COROL.

By reasoning in a similar manner, the value of an annuity, to continue as long as any two (out of four lives) shall be in being, may be found.

$$\text{For } \frac{n-1}{n^2} \times 1 - \frac{n-1}{n} \left. \begin{array}{l} \text{will be the probability of the} \\ \text{continuance, for} \\ \text{one year, of} \end{array} \right\} \begin{array}{l} \text{all four lives,} \\ \text{any 3 of them} \\ \text{any 2 of them} \end{array}$$

But because there are four combinations of three, and six combinations of two, in four; therefore the second probability must be taken four times, and the third six times.

And the probability of receiving the first year's rent will be

$$\frac{n-1^4}{n^4} + \frac{4 \times n-1^3}{n^3} \times 1 - \frac{n-1}{n} + \frac{6 \times n-1^2}{n^2} \times 1 - \frac{n-1}{n}$$

$$\text{But } \frac{n-1^3}{n^3} \times 1 - \frac{n-1}{n} = \frac{n-1^3}{n^3} - \frac{n-1^4}{n^4}$$

$$\text{And } \frac{n-1^2}{n^2} \times 1 - \frac{n-1}{n} = \frac{n-1^2}{n^2} - \frac{2 \times n-1^3}{n^3} + \frac{n-1^4}{n^4}$$

$$\text{Whence } \frac{6 \times n-1^2}{n^2} - \frac{12 \times n-1^3}{n^3} + \frac{6 \times n-1^4}{n^4}$$

$$+ \frac{4 \times n-1^3}{n^3} - \frac{4 \times n-1^4}{n^4}$$

$$+ \frac{n-1^4}{n^4}$$

$$\text{That is } \frac{6 \times n-1^2}{n^2} - \frac{8 \times n-1^3}{n^3} + \frac{3 \times n-1^4}{n^4} \text{ will be the pro-}$$

bability of receiving the first year's rent; and the value of the annuity may be found by adding six times the value of two joint lives to three times that of four joint lives; and subtracting eight times the value of three joint lives from the sum.

$$\text{Now six times the value of two joint lives is } \left. \begin{array}{l} 6P - \frac{12}{n} Q + \frac{6 \times 1-P}{n^2} R, \end{array} \right\}$$

three

$$\left. \begin{array}{l} \text{three times} \\ \text{that of four} \\ \text{joint lives} \end{array} \right\} 3P - \frac{12}{n} Q + \frac{18}{nn} R - \frac{12}{n^3} S + \frac{2 \times 1 - p}{n^4} T;$$

$$\text{their sum} \quad 9P - \frac{24}{n} Q + \frac{24 - 6p}{nn} R - \frac{12}{n^3} S + \frac{3 \times 1 - p}{n^4} T;$$

$$\left. \begin{array}{l} \text{eight times} \\ \text{that of 3} \\ \text{joint lives} \end{array} \right\} 8P - \frac{24}{n} Q + \frac{24}{nn} R - \frac{8 \times 1 - p}{n^3} S,$$

$$\text{Remains} \quad P - \frac{6p}{nn} R - \frac{4 + 8p}{n^3} S + \frac{3 \times 1 - p}{n^4} T, \text{ the value of the annuity.}$$

And thus we might proceed to find the value of an annuity, to continue as long as any k lives (out of m lives) shall be in being; but enough has been said to enable the reader to perform this, if he has leisure and inclination; therefore we proceed to things of more use.

QUESTION LXXXVIII.

It is required to find the value of the reversion of an estate, in fee simple, after a single life of a given age, allowing compound interest at a given rate,

SOLUTION.

From the value of the perpetuity (*viz.* $\frac{1}{r-1}$) subtract the value of the given life, found *per quest.* 68, or 73 (which, if

N 4

if the decrements of life are equal, will be

$\frac{1}{r-1} \frac{1-p \times r}{n \times r-1}$ and the remainder, (in that

case $\frac{1-p \times r}{n \times \frac{r-1}{2}}$) will be the value of the reversion re-

quired; which (if we put $\frac{r}{r-1} = Q$) will become $\frac{1-p}{n} Q$.

The above, expressed in words at length, follows:

The rule, for finding the value of the reversion of an estate of 1 l. per annum in fee simple, after a life of a given age, allowing compound interest, at a given rate, and supposing the decrements to be equal.

Let the number of years, which the given age wants of 86, be called the complement of life.

Seek in the tables for the present worth of 1 l. due at the end of the complement of life; subtract the number so found from unity; multiply the remainder by the number, which (in the table annexed to quest. 20) stands on n line with the given rate, under the letter Q ; and divide the product by the complement of life, so shall the quotient be the value of the reversion required.

E X A M P L E.

What is the value of the reversion of an estate in fee simple, of 1 l. per annum, after the life of a person of 54 years of age, allowing compound interest at 4 per Cent. ?

Here $(86-54=)$ 32 will be the complement of life,

0,285058 will be the present worth of 1 l. due at the end of 32 years.

And 650 will be the tabular number under Q .

Then

Then if $(1-0,285058=) 0,714942$ be multiplied by 650, the product will be 464,712 ; which, divided by 32, will quote 14,522 the value of the reversion.

QUESTION LXXXIX.

If it is required to find the present worth of 1 l. which is not to be received until the death of a person of a given age, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

SOLUTION.

One pound, or any other sum of money, may be conceived as the present worth of an estate in fee simple, equal to the interest of that sum for one year ; and consequently the reversion of a fee simple equal to that interest, after the given life, will be the value required.

Now (retaining the same symbols as in quest. 88), the interest of 1 l. for 1 year will be $r-1$; & if $\frac{1-p \times r}{n \times (r-1)}$ be mul-

tiplied by $r-1$, the product $\frac{1-p \times r}{n \times r-1}$ or $\frac{1-p \times r}{n} P$, will be the value required.

Which, expressed in words at length, follows :

The rule for finding the present worth of 1 l. due on the decease of a person of a given age, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, be called the complement of life ; and let one pound, and the interest for one year, be called the rate.

Seek in the tables for the present worth of one pound due at the end of the complement of life; subtract the number so found from unity; multiply the remainder by the rate, and that product by the number, which (in the table annexed to question 20) stands on a line with the given rate, under the letter P.

Divide the last product by the complement of life, so shall the quotient be the present worth required.

E X A M P L E.

A, who is heir to a considerable fortune upon the death of B, aged 54 years, would borrow a sum of money to be repaid, at that time, by him or his successors; how much ought he to receive now, for every pound that is then to be paid, allowing compound interest at four per Cent.?

Here $(86 - 54 =) 32$ will be the complement of life,
 $0,285058$ will be the present worth of one pound due at the end of 32 years,

And 25 will be the tabular number under P.

Then if $(1 - 0,285058 =) 0,714942$ be multiplied by 1,04 (the rate) it will produce 0,74354; and if this be multiplied by 25, the product will be 18,5885; which, being divided by 32, will give 0,5809, for the present worth required.

C O R O L.

In the same manner, the present worth of one pound, to be received at the expiration of any number of lives, may be found from the solutions of questions 89, 90, &c. by multiplying their result by the interest of one pound for one year.

Or,

Or, if the value of the given life or lives, be known: then, multiply it by the interest of one pound for one year; and subtract the product from unity, and the remainder will be the present worth required.

In the above example, the given life is worth 10,478; which, being multiplied by 0,04, produces ,41912; and this, being subtracted from unity, leaves ,58088, as above.

SCHOLIUM.

To obviate any doubt, that may arise in the reader's mind concerning the truth of the principle on which this question has been solved; it is thought proper to insert the following process; which is performed in the manner frequently used by Mr. *De Moivre*.

Let N represent the value of an annuity on the given life; and let us suppose it to be equal to an annuity for a number of years certain, suppose m ; then it will be evident, that the proposed sum will not be due till the expiration of those m years; and consequently one pound then due will be worth but $\frac{1}{r^m}$; by question 144, part 2, and vol. I.

Now the present value of an annuity of one pound to continue m years, is $\frac{r^m - 1}{r^m \times r - 1}$ (by question 149, part 2, vol. I.) which expression is supposed to be equal to the value of the life.

That is $\frac{r^m - 1}{r^m \times r - 1} = N$; or $r^m - 1 = r - 1 \times N r^m$;

Th. $r^m - r - 1 \times N r^m = 1$; or $r^m = \frac{1}{1 - r - 1 \times N}$;

Consequently $\frac{1}{r^n} = \left(\frac{1-r-1 \times N}{1} \right) \frac{1}{1-r-1} \times N$

which is the last given rule.

But $N = \frac{1}{r-1} - \frac{1-p \times r}{n \times r-1}$ by quest. 56;

Th: $r-1 \times N = 1 - \frac{1-p \times r}{n \times r-1}$;

And $1-r-1 \times N = \frac{1-p \times r}{n \times r-1}$ which is the same expression as before.

In like manner, if it were required to know what sum ought to be paid, on the death of a person of a given age, in consideration of one pound now received, the same will appear to be $\frac{n \times r-1}{1-p \times r}$.

For the amount of one pound at compound interest for the time n , is $r^n = \frac{1}{1-r-1 \times N}$, by the above process:

Also $1-r-1 \times N = \frac{1-p \times r}{n \times r-1}$; therefore $r^n = \frac{n \times r-1}{1-p \times r}$

$$\frac{n}{1-p \times r P}$$

The rule, in words at length, given in page 274, will serve for this purpose; if, instead of the last paragraph, you read thus

Divide the complement of life by the last found product, and the quotient will be the sum, which ought to be paid at the expiration of the given life.

Note, the symbol (N) may be above expounded by the value of any number of lives; when such value enters the question, instead of the value of a single life.

Q U E S.

QUESTION XC.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an estate in fee simple, after the longest liver of two persons of given ages?

SCHOLIUM.

If the same symbols be retained as in quest. 76, then if from P , the value of the perpetuity, be subtracted the value of an annuity for the longest of the two lives, the remainder, *viz.*

$$\frac{rPP}{m} \times \frac{1-p \times r + 1 \times P}{n} - p + \frac{1}{r^m}, \text{ will be the value required.}$$

Or, by the approximation in quest. 77, the value will be

$$P - M - N - \frac{n-1}{6r} \times \frac{n+1}{2m}.$$

The numerical process, and rules in words at length, are omitted in this, and some of the following questions, which will but rarely occur in practice; because they differ very little from the processes, and rules given in the questions, in this and those following, quoted: To instance in this, if the result of the approximation in quest. 77, be taken from the number, which (in the table annexed to quest. 20) stands on a line with the given rate, under the letter P ; the remainder will be the value of the reversion.

Also, if the given ages are equal, then the value of the reversion will (*per Cent. 78*) be

$$R \times \frac{1-p}{mn} - 2 \times \frac{2p}{n}, \text{ Or } R \times \frac{1-p}{n} - 2p \times \frac{1}{n}.$$

And

And the approximation thereto, will (*per* quest. 79) be

$$P - \frac{1}{2} \times \frac{3n+1 \times N}{n} - \frac{n-1}{6r}$$

QUESTION XC.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an estate, in fee simple, after the longest liver of three persons of given ages.

SOLUTION.

If the same symbols be retained as in quest. 80, then, if from P , the value of the perpetuity, be subtracted the value of an annuity, on the longest of the three given lives, the remainder will be the value of the reversion required, *viz.*

$$\frac{rP^2}{2} \times \left\{ \begin{array}{l} + \frac{1-r \times r + 2^2 - 3 \times P^2}{nm} \\ - \frac{\frac{n}{m}p + \frac{1}{r^2} + \frac{2}{r^m}}{2p + \frac{1}{r^m} \times \frac{r+1 \times P}{m}} \end{array} \right.$$

And the approximation thereto will (*per* quest. 81) be

$$P - F - \frac{1}{2r} \times N - \frac{n-1}{6r} \times \frac{n+1}{2m} + M - \frac{m-1}{6r} \times m + \frac{1}{r}$$

Also,

Also, if the three ages be equal, then the value of the reversion will (*per* quest. 85) be

$$\frac{1-p}{n^3} S - \frac{3p}{nn} R - \frac{3p}{n} Q.$$

And the approximation thereto will (*per* scholium to the same quest) be

$$P - N - \frac{3n+1 \times n+1}{4nn} \times N - \frac{n-1}{6r}.$$

COROL.

Since the reversion of an estate, in fee simple, after the longest of

$$\left. \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right\} \text{equal lives, is } \left\{ \begin{array}{l} \frac{1-p}{n} Q; \\ \frac{1-p}{nn} R - \frac{2p}{n} Q; \\ \frac{1-p}{n^3} S - \frac{3p}{nn} R - \frac{3p}{n} Q; \end{array} \right.$$

Therefore, the reversion of an estate, in fee simple, after the longest of m lives, will be

$$-\frac{m}{n} Q - \frac{m \cdot m - 1 \cdot p}{1 \cdot 2 \cdot nn} R - \frac{m \cdot m - 1 \cdot m - 2 \cdot p}{1 \cdot 2 \cdot 2 \cdot n^3} S(m-1) + \frac{1-p}{n^m} Z.$$

Where Z denotes the $m+1$ factor in the scale, P , Q , R , &c. (quest: 20).

SCHOLIUM.

The questions relating to the reversion of an estate, in fee simple, after two or more unequal joint lives, are omitted, because it is supposed that the resolution thereof will hardly ever occur in practice; And because the principle, on which their solution depends, is the same with the former,

mer, viz. the taking the value of the lives from the perpetuity.

Whence, and from corol. to quest 75, it will appear, that the value of the reversion of an estate, in fee simple, after

$$\left. \begin{array}{l} 2 \\ 3 \\ 4 \end{array} \right\} \begin{array}{l} \text{joint lives} \\ \text{will be} \end{array} \left\{ \begin{array}{l} \frac{2}{n} Q - \frac{1-p}{n^2} R \\ \frac{3}{n} Q - \frac{3}{n^2} R + \frac{1-p}{n^3} S \\ \frac{4}{n} Q - \frac{6}{n^3} R + \frac{4}{n^3} S - \frac{1-p}{n^4} T \end{array} \right.$$

&c.

And therefore the value of the reversion of an estate, in fee simple, after m joint lives, will be

$$\frac{m}{n} Q - \frac{m \cdot m - 1}{n \cdot 2n} R + \frac{m \cdot m - 1 \cdot m - 2}{n \cdot 2n \cdot 3n} S (m-1) + \frac{1-p}{n^m} Z$$

In which expression, Z signifies the $m+1$ term of the series of factors P, Q, R , &c. (see quest. 20) and the last term will be affirmative, when the term immediately preceding is negative; and negative, when that is affirmative.

QUESTION XCII.

There is a leasehold estate of one pound *per annum*, to continue n years; which A (whose complement of life is m) is to enjoy, if he lives so long; but if he dies before the expiration of the said n years, then B and his heirs are to have the remainder thereof; what is the value of B 's interest therein, supposing the decrements of life to be equal?

SOLU-

SOLUTION.

From $1 - \frac{1}{r^n} \times \frac{1}{r-1}$ the value of the annuity for n years certain (quest. 15), subtract

$$1 - \frac{1}{r^n} \times \frac{1}{r-1} + \frac{n}{mr^n \times r-1} - 1 - \frac{1}{r^n} \times \frac{r}{m \times r-1}$$
 the pre-

sent worth of the annuity for n years, if A shall live so long (quest. 57), and the remainder viz.

$$1 - \frac{1}{r^n} \times \frac{r}{m \times r-1} - \frac{n}{mr^n \times r-1}; \text{ Or, (putting } P = \frac{1}{r-1}$$

$$\text{and } Q = \frac{r}{r-1}) 1 - \frac{1}{r^n} \times \frac{Q}{m} - \frac{nP}{mr^n}$$

$$\text{That is } \frac{Q}{m} - \frac{Q}{mr^n} - \frac{nP}{mr^n}, \text{ Or } \frac{Q}{m} - \frac{Q+nP}{mr^n};$$

Or (putting $\frac{1}{r^n} = P$) $\frac{Q-Q+nP \times P}{m}$ will be the value required.

Which, expressed in words at length, follows:

The rule for finding the value of the reversion of a leasehold estate of one pound per annum, of which a given number of years remain unexpired, after the decease of a person of a given age; allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, be called the complement of life.

Multiply that number which (in the table annexed to quest. 20) stands, on a line with the given rate, under the letter P , by the number of years unexpired in the lease; to the product add the number, standing under the letter Q , in the same table; and multiply the sum, by the present worth of one pound due at the expiration of the lease.

Subtract

(by writing as before Q for $\frac{r}{r-1}$) for the whole value of B 's interest.

C O R O L

If the question had proposed to have found the value of this reversion from tables of observations; then the value of an annuity for x years, if A shall live so long (which may be found by quest. 60) being taken from the value of an annuity for n years certain; the difference will shew B 's interest for the first x years: also if a and b be, severally, the numbers proportional to the living of A 's age, and at the end of x years, then $\frac{b}{a \times r^n \times r - 1}$ will be the remaining part of his interest for ever.

The above rule, expressed in words at length, follows:

The rule for finding the value of the reversion of an estate of any kind for annuity, in fee simple, after the death of a minor (if that happen before he comes to be of age to possess it) allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the minor wants of 86, be called his complement of life; and, which he wants of being at age, his complement of possession.

From unity subtract the present worth of one pound due at the end of the complement of possession; multiply the remainder by the number which (in the table annexed to quest. 20) stands, on a line with the rate, under the letter Q ; and divide the product by the complement of life, so shall the quotient be the value of the reversion required.

E X A M.

E X A M P L E.

Suppose that *B* and his heirs are entitled to an estate of one pound *per annum*, if *A* (who is 10 years of age) should die before he is 21; what is the reversion worth, allowing compound interest at four *per Cent*.

Here $(86-10=)$ 76 will be the complement of life,

And $(21-10=)$ 11 that of possession.

Also the present worth of one pound due at the end of 11 years is 0,6496;

And the number standing, on a line with the rate, under \mathcal{Q} , is 650.

Now if $(1-0,6496=)$ 0,3504, be multiplied by 650, the product will be 227,76; which, being divided by 76, will quote 2,997, the value of the reversion required.

Q U E S T I O N . X C I V .

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the decease of the present possessor, whose age is also given?

S O L U T I O N .

Let the complement of the life of the present possessor be denoted by *n*, and that of the expectant in reversion by *m*.

Then

Then the probabilities of the possessor's dying, in the first, second, third, &c. years, will be

$$1 - \frac{n-1}{n}, 1 - \frac{n-2}{n}, 1 - \frac{n-3}{n} \text{ \&c.}$$

And the probability of the expectant's living to the end of the first, second, third, &c. years, will be

$$\frac{m-1}{m}, \frac{m-2}{m}, \frac{m-3}{m} \text{ \&c.}$$

And since these two events are independent on each other, it will follow (from quest. 28) that,

$\frac{m-1}{m} \times 1 - \frac{n-1}{n}$; $\frac{m-2}{m} \times 1 - \frac{n-2}{n}$ &c. will be the probabilities of the expectant's receiving of the first, second, &c. yearly payment;

Which probabilities, being taken as the values of the first, second, &c. payments; being expanded by multiplication, and their present values being found, will give the value of the reversion, *viz.*

$$\frac{m-1}{mr} - \frac{m-1 \times n-1}{mar} + \frac{m-2}{mr^2} - \frac{m-2 \times n-2}{mnr^2} \text{ \&c.}$$

Now $\frac{m-1}{mr} + \frac{m-2}{mr^2} + \frac{m-3}{mr^3}$ &c. is the value of an annuity on the life of the expectant;

And $\frac{m-1 \times n-1}{mnr} + \frac{m-2 \times n-2}{mnr^2} + \frac{m-3 \times n-3}{mnr^3}$ &c. is the value of an annuity, to continue during the joint lives of the possessor and expectant.

Therefore, if from the value of the expectant's life, the value of the joint lives of the possessor and expectant be taken, the remainder will be the value of the reversion:

Or, if from the value of the longest of the two lives, the life of the present possessor be taken; the remainder will be the value of the reversion.

Now

Now the value of the longest of the two lives (per quest.

$$76) \text{ is } P + Q \times \frac{1}{m^m} + \frac{1}{m^n} - R \times \frac{1-p}{mn};$$

And the value of the possessor's life (per quest. 56) is

$$P + Q \times \frac{1}{n} + \frac{1}{nr^n};$$

The difference of which will be

$$Q \times \frac{1}{n} + \frac{1}{nm} + \frac{1}{nr^n} - \frac{1}{mn} - R \times \frac{1-p}{mn}, \text{ the value of the reversion.}$$

Which expression (restoring rPP , for Q ; $r+1$ for P , for R , and p for $\frac{1-p}{n}$) will become

$$rPP \times m + \frac{n}{r^m} + np - mp \times \frac{1}{nm} - P^2 \times r+1 \times \frac{1-p}{mn},$$

$$\text{Or } \frac{rPP}{nm} \times m + \frac{n}{r^m} - m - n \times p - r+1 \times P \times 1-p.$$

EXAMPLE.

Let the expectant be 43 years of age, and the possessor 54 years, allowing compound interest at four per Cent.

Then $(86-43=) 43=m$; $(86-54=) 32=n$; $p=$

$$0,285058; \frac{1}{r^m} = 0,185168; r=1,04; r+1=2,04 \text{ \&}$$

$$P=25; 32 \times 0,185188=5,925376; m-n \times p=3,135638;$$

$$1-p \times r+1 \times P=36,462050;$$

$$\text{Then } 43 + 5,925378 - 3,135638 - 36,462050 = 9,327688;$$

$$\text{And } \frac{rPP}{nm} = \frac{650}{1376}, \text{ Th. } \frac{650}{1376} \times 9,327688 = 4,406.$$

Which

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Which is the same answer, as will arise from the process first above directed, &c.

From the value of the expectant's life, which, $\left\{ \begin{array}{l} \text{exam. 1, quest. 56, is} \\ 12,683, \end{array} \right.$

Take the value of the joint lives, which, by $\left\{ \begin{array}{l} \text{exam. 1, quest. 64, is} \\ 8,277; \end{array} \right.$

Remains the value of the reversion - - - 4,406.

Or, if the approximation to the value of the joint lives be made use of, then

From the value of the expectant's life M ,

Take the value of the joint lives, viz. $M - N - \frac{n-1}{6r} \times \frac{n+1}{2m}$

The remainder, $M - N + N - \frac{n-1}{6r} \times \frac{n+1}{2m}$, will be the value of the reversion.

Which, in words at length, follows:

The rule for finding the value of the reversion of an annuity, for a given life, after the failure of another given life, elder than the former; having the values of the single lives given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given lives want of 86, be called their complements; and let the sum of one pound and its interest for one year, be called the rate.

From the lesser complement subtract one, and divide the remainder by six times the rate, or find this quotient in the last table.

Subtract this quotient from the value of the possessor's life; multiply the remainder by the lesser complement more one; and divide the product by twice the greater complement.

Lastly, from the value of the life of the expectant take the value of the life of the possessor; and to the remainder, add the

the last found quotient ; so shall the sum be the value of the reversion required.

EXAMPLE.

B, who is 54 years of age, is possessed of an annuity of one pound, which, after his decease, is to descend to *A*, aged 43 years, if he survives *B*, for the remaining part of his life; what is the present value of *A*'s interest in the annuity, allowing compound interest at four per Cent.?

Here the value of *A*, the expectant's life, is 12,683,

And the value of *B*, the possessor's life, is 10,478;

Also $(86-43=)$ 43 is the greater complement,

$(86-54=)$ 32 is the lesser complement ;

And $(1+1,04=)$ 1,04 is the rate.

If $(32-1=)$ 31 be divided by $(6 \times 1,04=)$ 6,24, the quotient will be 4,970.

Then $(10,478-4,970=)$ 5,508, being multiplied by $(32+1=)$ 33, produces 181,764 ; which, divided by $(43 \times 2=)$ 86, quotes 2.114.

Lastly, $(12,683-10,478=)$ 2,205; and $(2,205+2,114=)$ 4,319 will be the value of the reversion required.

CASE 2. When the possessor is younger than the expectant, then the complement of the expectant's life $=n$, and the possessor's $=m$, the value of the possessor's life will be,

$$P + 2 \times \frac{1}{m} + \frac{1}{mn};$$

Which, taken from the value of the longest of the two lives, will leave $2 \times \frac{1}{m} + \frac{1}{mn} - R \times \frac{1-p}{mn}$, the value of the reversion ;

Which (writing p for $\frac{1}{n}$, and rPP for Q , &c.) will be-

$$\text{come, } 1+p-\frac{1-p \times r+1 \times P}{n} \times \frac{PPr}{m}.$$

For example, let the expectant be 54 years of age, and the possessor 43 years, allowing compound interest at four per Cent.

Then $m=43$; $n=32$; $p=.285058$; $r=1.04$; and

$$P=.25; \quad 1-p \times r+1 \times P=.36,46205; \quad \text{and } \frac{36,46205}{32} =$$

$$1,139438 :$$

Now $(1,285058-1,139438) \times \frac{650}{43} = 2,2012$
the value of the reversion required.

Or, if the approximation to the value of the joint lives be made use of, then,

From the value of the expectant's life N ,

Take the value of the joint lives, viz.

$$N-N-\frac{n-1}{6r} \times \frac{n+1}{2m};$$

The remainder, $N-\frac{n-1}{6r} \times \frac{n+1}{2m}$, will be the value of the reversion required.

Which, expressed in words at length, follows :

The rule for finding the reversion of an annuity, for a given life after the failure of another given life, younger than the former; having the value of the elder life given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given lives want of 86, be called their complements; and let the sum of one pound, and its interest for one year, be called the rate.

From

From the lesser complement subtract one, and divide the remainder by six times the rate, or find the quotient in the last table.

Subtract this quotient from the given value of the elder life; multiply the remainder by the lesser complement more one; and divide the product by twice the greater complement; so shall the quotient be the value of the annuity required.

EXAMPLE.

A, who is 43 years of age, is possessed of an annuity of one pound, which, after his decease, is to belong to B, who is 54 years old (if he survives A) for the remaining part of his life; what is the present value of B's interest in the annuity, allowing compound interest at four per Cent.?

Here the value of B the elder life is 10,478;

$(86-43=)$ 43, will be the greater complement

$(86-54=)$ 32, lesser

And $(1+04=)$ 1,04, the rate

Now $(32-1=)$ 31, being divided by $(6 \times 1,04=)$ 6,24; the quotient will be 4,970.

Then $(10,478-4,970=)$ 5,508 being multiplied by $(32+1=)$ 33, produces 181,764; which, divided by $43 \times 2=$ 86, quotes 2,114, the value of the reversion required.

CASE 3. When the possessor and expectant are of equal ages, then the value of the reversion will be

$$\frac{1}{1+p} - \frac{1-p \times r + 1 \times P}{n} \times \frac{PP_r}{n}, \text{ Or } \frac{1+p}{n} \underline{Q} - \frac{1-p}{nn} R.$$

And the approximation will be

$$N - \frac{n-1}{Or} \times \frac{n+1}{2n}; \text{ as will appear by writing } n \text{ for } m \text{ in}$$

the expressions obtained in the last case.

The process, being the same, in both methods, with that given in the example to the last case (excepting only that the complements are equal) an example would be superfluous.

QUESTION XCV.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the joint lives of two persons of given ages, after the decease of the present possessor, whose age is also given.

SOLUTION.

Let the complement of the life of the present possessor be denoted by n , and those of the expectants by t and m ;

Then the probability of the possessor's dying, in the first, second, or third, &c. years, will be

$$1 - \frac{n-1}{n}, 1 - \frac{n-2}{n}, 1 - \frac{n-3}{n} \text{ \&c.}$$

And the probability of the joint continuance of the two expectants lives will, for the first, second, third, &c. years, be

$$\frac{m-1 \times t-1}{mt}, \frac{m-2 \times t-2}{mt}, \frac{m-3 \times t-3}{mt} \text{ \&c.}$$

Which probabilities, being severally multiplied into the corresponding probability of the possessor's dying, will give the values of the first, second, &c. payments of the reversion, *viz.*

$$\frac{m-1 \times t-1}{mt} - \frac{m-1 \times t-1 \times n-1}{min} + \frac{m-2 \times t-2}{mt} - \left(\frac{m-2 \times t-2 \times n-2}{min}, \&c. \right)$$

the present values of which payments will be

$$\frac{m-1 \times t-1}{mtr} - \frac{m-1 \times t-1 \times n-1}{minr} + \frac{m-2 \times t-2}{mtr^2} - \left(\frac{m-2 \times t-2 \times n-2}{minr^2}, \&c. \right)$$

Now $\frac{m-1 \times t-1}{mtr} + \frac{m-2 \times t-2}{mtr^2} + \frac{m-3 \times t-3}{mtr^3}, \&c.$ is

the value of an annuity on the joint lives of the two ex-

pectants, and $\frac{m-1 \times t-1 \times n-1}{minr} + \frac{m-2 \times t-2 \times n-2}{minr^2}, \&c.$

is the value of an annuity on the joint lives of the possessor and two expectants.

Therefore ; if, from the value of the joint lives of the two expectants, be taken the value of the three joint lives of the possessor and two expectants, the remainder will be the reversion of the two joint lives after one.

The application of the above expressions of the values of joint lives to this rule admits of eight cases.

CASE 1. When the possessor is older than either of the expectants, their ages being unequal.

Then the value of an annuity on the joint lives of the two expectants will be

$$P + \frac{1}{m^m} - \frac{1}{t^m} - \frac{1}{m} - \frac{1}{t} \times 2 + \frac{1}{mt} - \frac{1}{mtr^m} \times R;$$

and the value of any annuity on three joint lives will be

$$P + \frac{1}{nr^n} + \frac{n}{mtr^n} - \frac{1}{tr^n} - \frac{1}{mtr^n} - \frac{1}{t} - \frac{1}{m} - \frac{1}{n} \times 2$$

$$+ \frac{1}{mt} + \frac{1}{nt} + \frac{1}{nm} - \frac{1}{ntr^n} + \frac{2}{mtr^n} - \frac{1}{nmr^n} \times R - \frac{1-p}{nm} S;$$

Therefore the difference of those two expressions, viz.

$$\frac{1}{mtr^n} - \frac{1}{tr^n} - \frac{1}{nr^n} - \frac{n}{mtr^n} + \frac{1}{tr^n} + \frac{1}{mtr^n} + \frac{1}{n} \times 2$$

$$+ \frac{1}{ntr^n} + \frac{1}{nmr^n} - \frac{1}{mtr^n} - \frac{2}{mtr^n} - \frac{1}{nt} - \frac{1}{nm} + R$$

$$+ \frac{1-p}{nm} \times S \text{ will be the value of the reversion; which}$$

may be wrote as follows:

$$\frac{m-1-n \times m-n}{m} \times \frac{1}{r^n} + \frac{m-1-n}{m} \times \frac{n}{r^n} \times \frac{2}{nm} =$$

$$m + 1 - m + 1 - 2n \times \frac{1}{r^n} + n \times \frac{1}{r^n} \times \frac{R}{nm} + \frac{1-p}{nm} \times S$$

SCHOLIUM.

If the approximations to the values of the joint lives be applied to the solution of this question; then, in this case,

The value of the expectants joint lives will (by schol.

to quest. 64) be $M \times 1 - \frac{m+1}{2t} + \frac{m+1 \times m-1}{12tr}$ and the

value of the three joint lives will be, by schol. quest. 69.

$$N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} + \frac{n+1}{2m} \times \frac{n-1}{6r} \times \frac{3m+2t-n-3}{2t};$$

which expression, if subtracted from the former, will leave a remainder, equally difficult to be applied to numbers, as the two separate expressions are; which subtraction is therefore omitted.

Now

Now by *examp. 1. schol. quest. 64*, the two joint lives are worth 8,364.

And by *schol. quest. 69*, the three joint lives are worth 5,199.

Therefore the reversion will be 3,165.

CASE 2. When the expectants are of equal ages, and the possessor elder than either of them ;

The solution of this may be deduced from that of case 1, by making $k=m$ as follows.

$$\frac{mm-m-n}{r^n} \times \frac{1}{r^n} \times \frac{2}{nmn} - \frac{2m-m-n}{r^n} \times \frac{2}{r^n} + \frac{n}{r^n} \times \frac{R}{nmn} \left(+ \frac{1-p}{nmn} \times S \right)$$

The approximations, to the values of the joint lives, are as follow :

The two equal joint lives *per schol. to quest. 65*, will

be $M \times \frac{m-1}{2m} + \frac{m+1 \times m-1}{12mr}$ and the value of the three joint lives, the two younger of which are equal, will be

$$N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2m} + \frac{n-1}{6r} \times \frac{n+1}{2m} \times \frac{4m-n-3}{2m}$$

by *quest. 70*. But in this case also the subtracting the latter expression from the former will not much shorten the numerical process.

CASE 3. When the possessor is younger than either of the expectants, their ages being unequal.

Here we must call the complement of the possessor's life r , and those of the expectants m and n .

Then the value of the joint lives of the expectants will be

$$P + \frac{1}{nt^n} - \frac{1}{mt^n} - \frac{1}{n} - \frac{1}{m} \times Q + \frac{1}{nm} - \frac{1}{mnt^n} \times R$$

From which, the value of the three joint lives being taken, will leave

$$\frac{1}{nt^n} + \frac{1}{t} - \frac{n}{mt^n} \times Q +$$

$$\frac{1}{nt^n} - \frac{2}{mt^n} - \frac{1}{mt} - \frac{1}{nt} \times R + \frac{1-p}{nmt} S$$

the value of the reversion, which may be reduced to

$$mn + m - n \times n \times \frac{1}{n} \times \frac{Q}{nmt} - n + m + 2n - m \times \frac{1}{n} \times \frac{R}{nmt} + \frac{1-p}{nmt} \times S$$

Now if the approximations to the values of the joint lives be applied to this case: then the joint lives of the expectants will (*per schol.* to quest. 64) be

$$N \times 1 - \frac{n+1}{2m} + \frac{n+1 \times n-1}{12rm}; \text{ and the value of the three}$$

$$\text{joint lives } N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} + \frac{n+1}{2m} \times \frac{n-1}{2r} \times \frac{2m+2t-n-3}{6t}, \text{ which may be wrote}$$

$$N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} + \frac{n+1 \times n-1}{12rm} \times \frac{2m+2t-n-3}{2t};$$

which last expression, being subtracted from the value of the joint lives of the expectants, will leave

$$N \times 1 - \frac{n+1}{2m} \times 1 - 1 - \frac{n+1}{2t} + \frac{n+1 \times n-1}{12rm} \times \left(1 - \frac{2m+2t-n-3}{2t} \right)$$

But

$$\text{But } 1 - 1 - \frac{n+1}{2t} = \frac{n+1}{2t}, \text{ \& } 1 - \frac{2m+2t-n-3}{2t} = \frac{2m-n-3}{2t}$$

whence the value of the reversion will be

$$N \times 1 - \frac{n+1}{2m} \times \frac{n+1}{2t} - \frac{n+1 \times n-1}{12rm} \times \frac{2m-n-3}{2t},$$

$$\text{Or } N \times 1 - \frac{n+1}{2m} - \frac{n-1 \times 2m-n-3}{12rm} \times \frac{n+1}{2t} : \text{ which,}$$

$$\text{because } 1 - \frac{n+1}{2m} = \frac{2m-n-1}{2m} \text{ will become}$$

$$N \times \frac{2m-n-1}{2m} - \frac{n-1 \times 2m-n-3}{12rm} \times \frac{n+1}{2t}$$

But, since $2m-n-3$ differs but 2 from $2m-n-1$, and the divisor $12rm \times 2t$ is great in comparison of the difference ; let $2m-n-1$ be wrote for $2m-n-3$, and the reversion will be

$$\left(N \times \frac{2m-n-1}{2m} - \frac{n-1}{6r} \times \frac{2m-n-1}{2m} \times \frac{n+1}{2t} \right) \Delta - \frac{n-1}{6r} \left(\times \frac{n+1 \times 2m-n-1}{4mt} \right)$$

Which, expressed in words at length, will stand as follows :

The rule to find the value of the reversion of an annuity, on two joint lives after one ; when the expectants are severally elder than the possessor, and of unequal ages, having the value of the oldest life given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life; and let the sum of one pound, and its interest for one year, be called the rate.

From the complement of the elder expectant's life subtract one, and divide the remainder by six times the rate, or let this quotient be found by table the last; let the quotient be taken from the value of the eldest life, calling the remainder R .

From twice the complement of the younger expectant's life, take the complement of the elder expectant and one; multiply the remainder by the complement of the elder expectant more one; and that product by the above found remainder R ; divide this last product by four times the product of the complements of the possessor and younger expectant, so shall the quotient be the value of the reversion required.

EXAMPLE.

If the reversion of an annuity for two joint lives of 54 and 66, after a life of 43, allowing 4 l. per Cent. be required,

Then 7,333 will be the value of the eldest life;

(86—43=) 43 will be the complement of the possessor's life,

(86—54=) 32 that of the younger expectant,

(86—66=) 20 that of the elder expectant,

And (1+ .04=) 1.04 the rate.

Then (20—1=) 19 being divided by (6×1.04=) 6.24 will quote 3.045; which, taken from 7,333, will leave 4,288 for the remainder R .

And (2×32—20—1=64—21=) 43 being multiplied by (20+1=) 21, will produce 903; which being also multiplied by (R =) 4,288 produces 3872,064; which last product

product being divided by $(4 \times 32 \times 43 =) 5504$, will quote 0.7035, the value of the reversion required.

CASE 4. When the expectants are of equal ages, and the possessor younger than either of them.

The solution of this case may be deduced from that of the former, by making $m=n$, as follows :

$$\frac{R}{t} - 2n + \frac{n}{r^n} \times \frac{R}{nrt} + \frac{1-p}{nrt} S$$

Now the approximation to the value of the two equal joint lives of the expectants will (*per* quest. 65) be

$$N \times \frac{n-1}{2n} + \frac{n+1 \times n-1}{12rn}; \text{ and the value of the three joint}$$

$$\text{lives } N \times \frac{n-1}{2n} \times \frac{2t-n-1}{2t} + \frac{n+1 \times n-1}{12rn} \times \frac{2t+n-3}{2t};$$

(*by* quest. 71).

Whence, by subtraction, the value of the reversion will be

$$N \times \frac{n-1}{2n} \times 1 - \frac{2t-n-1}{2t} + \frac{n+1 \times n-1}{12rn} \times 1 - \frac{2t+n-3}{2t}$$

$$\text{But } 1 - \frac{2t-n-1}{2t} = \frac{n+1}{2t} \text{ and } 1 - \frac{2t+n-3}{2t} = \frac{n-3}{2t};$$

Therefore the value of the reversion will become.

$$N \times \frac{n-1}{2n} \times \frac{n+1}{2t} - \frac{n+1 \times n-1}{12rn} \times \frac{n-3}{2t};$$

$$\text{Or } N - \frac{n-3}{6r} \times \frac{n+1 \times n-1}{4nt};$$

Also, because $(n+1 \times n-1) n-1$ differs from nn only by unity, and is to be divided by $4nt$, a number great in respect thereto; therefore let nn be wrote for $n+1 \times n-1$, and the value of the reversion will become,

$\left(N - \frac{n-3}{6r} \times \frac{nn}{4nt} \right) N - \frac{n-1}{6r} \times \frac{n}{4t}$ which, in words at length, follows :

The rule to find the value of the reversion of an annuity, on two equal joint lives after one, when the expectants are elder than the possessor ; having the value of the single life of one of the expectants given, allowing compound interest at a given rate ; and supposing the decrements of life to be equal.

Let the number of years, which each life wants of 86, be called their complements of life ; and let the sum of one pound, and its interest for one year, be called the rate.

From the complement of the expectant's life subtract three, and divide the remainder by six times the rate, or find the quotient by table the last.

Subtract this quotient from the given value of the expectant's life ; multiply the remainder by the complement of the expectant's life ; and divide the product by four times the complement of the possessor's life, so shall the quotient be the value of the reversion required.

EXAMPLE.

If the lives in reversion are each 66, the possessor 43, and interest four per Cent.

Then 7.333 will be the value of each expectant's life ;

(86—43=) 43, the comp. of the possessor's life,

(86—66=) 20, that of each expectants,

And (1+.04=) 1.04 the rate.

Now (20—3=) 17, being divided by (6×1.04=) 6.24, will quote 2,724.

And

And $(7,333-2,724=)$ 4,609, being multiplied by 20, produces 92,180; which, being divided by $(4 \times 43=)$ 172, will quote 0,536, the value of the reversion required.

CASE 5. When the expectants are, one elder, and the other younger, than the possessor.

Here the complement of the possessors life must be called m , and those of the expectants t and n .

Then the value of the joint lives of the expectants will be

$$P + \frac{1}{nr^n} - \frac{1}{tr^n} - \frac{1}{n} - \frac{1}{t} \times 2 + \frac{1}{nt} - \frac{1}{ntr^n} \times R$$

From which, taking the value of the three joint lives, there will remain

$$\frac{1}{mnr^n} + \frac{1}{m} - \frac{n}{mtr^n} \times 2 +$$

$$\left(\frac{1}{nmr^n} - \frac{2}{mtr^n} - \frac{1}{mt} - \frac{1}{mn} \times R + \frac{1-p}{nmt} S \right)$$

the value of the reversion, which may be reduced to

$$nt + t - n \times n \times \frac{1}{r^n} \times \frac{2}{nmt} - n + t + 2n - t \times \frac{1}{r^n} \times \frac{R}{nmt} + \frac{1-p}{nmt} S.$$

Now the approximation, to the value of the joint lives of the expectants, will be

$$N \times 1 - \frac{n+1}{2t} + \frac{n+1 \times n-1}{12rt} \text{ and the approximation to the}$$

value of the joint lives,

$$N \times 1 - \frac{n+1}{2m} \times 1 - \frac{n+1}{2t} + \frac{n+1 \times n-1}{12rt} \times \frac{2m+2t-n-3}{2m};$$

the difference of which will be

$$N \times 1 - \frac{n+1}{2t} \times \frac{n+1}{2m} - \frac{n+1 \times n-1}{12rt} \times \frac{2t-n-3}{2m}$$

$$\text{Or } N \times 1 - \frac{n+1}{2t} - \frac{n-1 \times 2t-n-3}{12rt} \times \frac{n+1}{2m}$$

Again, (because $1 - \frac{n+1}{2t} = \frac{2t-n-1}{2t}$) the above may

be wrote as below

$$N \times \frac{2t-n-1}{2t} - \frac{n-1}{6r} \times \frac{2t-n-3}{8t} \times \frac{n+1}{2m}, \text{ which (writing } 2t-n-1 \text{ for } 2t-n-3 \text{ as in case 3) will become}$$

$$N - \frac{n-1}{6r} \times \frac{n+1 \times 2t-n-1}{4mt}, \text{ the value of the reversion required.}$$

This may be expressed by the same words as in the rule for case 3; to which the reader is referred.

EXAMPLE.

If the expectants are severally 43, and 66 years of age; the possessor 54; and the rate of interest four *per Cent*.

Then 7,333 will be the value of the oldest life;

(86—54=) 32 will be the complement of the possessor's life,

(86—43=) 43 that of the younger expectant;

(89—66=) 20 that of the elder expectant,

And (1+1,04=) 1,04 the rate.

Then (20—1=) 19, being divided by (6×1,04=) 6,24, will quote 3,045; which, taken from 7,333, will leave 4,288, for the remainder *R*.

And (2×43—20—1=86—21=) 65, being multiplied by (20+1=) 21, will produce 1365; which, being also multiplied by the remainder (*R*) 4,288, produces 5853,120; which

which last product, being divided by $(4 \times 32 \times 43 = 5504$, will quote 1,063, the value of the reversion required.

CASE 6. When the elder expectant and the possessor are of the same age.

This case may be solved from the last, by writing n for m ; and the value of the reversion will be

$$nt + t - n \times \frac{n}{r} \times \frac{2}{nnt} - n + t + 2n - 1 \times \frac{1}{rn} \times \frac{R}{nnt} + \frac{1-p}{nnt} S;$$

And the approximation thereto $N - \frac{n-1}{6r} \times \frac{n+1 \times 2t - n-1}{4nt}$

which may be expressed by the same words, as the rule in case 3.

CASE 7. When the younger expectant and the possessor are of the same age.

This may be also solved from case 5, by writing m for t ; and the value of the reversion will be

$$nm + m - n \times \frac{n}{r} \times \frac{2}{nmm} - n + m + \frac{2n-m}{rn} \times \frac{R}{nmm} + \frac{1-p}{nmm} S$$

And the approximation thereto $N - \frac{n-1}{6r} \times \frac{n+1 \times 2m - n-1}{4mm}$

This is also expressible by the words given in the rule to case 3.

CASE 8. When the expectants and possessor are of equal ages.

Then calling the complement of their lives n , the solution may be obtained from any of the foregoing cases (suppose the first) by writing n for t and m ; also

$\frac{1}{r^n}$ for $\frac{1}{rn}$; as follows,

$$\frac{2}{n} - \frac{2+p}{2nr} R + \frac{1-p}{nrr} S.$$

Now

Now the approximation to the value of this reversion may be deduced from that in case 4, by writing n for t , viz.

$$N - \frac{n-3}{6r} \times \frac{n}{4n}, \text{ Or } N - \frac{n-3}{6r} \times \frac{1}{4} \text{ which, in words at}$$

length, follows :

The rule to find the value of the reversion of an annuity, on two equal joint lives after another life of the same age; having the value of a single life of that age given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which the given age wants of 86, be called the complement of life; and let the sum of one pound, and its interest for one year, be called the rate.

From the complement of life subtract three, and divide the remainder by six times the rate, or find this quotient by the last table.

Subtract this quotient from the given value of the single life; then shall one fourth part of the remainder be the value of the reversion required.

EXAMPLE.

If the given age be 66, and interest four per Cent. Then 7,333 will be the value of the single life,

(86—66=) 20 will be the complement of life,

And (1+ .04=) 1,04 the rate.

If (20—3=) 17, be divided by (6×1,04=) 6,24, the quotient will be 2,724.

And 7,333—2,724=4,609, the fourth part of which, viz. 1,152, will be the value of the annuity required.

QUEST.

QUESTION XCVI.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the joint lives of three persons of equal ages, after the decease of a fourth person of the same age?

SOLUTION.

Let the complement of their lives be denoted by n ;
Then the probability of the possessor's dying in the first, second, third, &c. year will be

$$1 - \frac{n-1}{n}, 1 - \frac{n-2}{n}, 1 - \frac{n-3}{n} \text{ \&c.}$$

And the probability of the joint continuances of the three expectants lives, will, for the first, second, third, &c. year be.

$$\frac{n-1}{n^3}, \frac{n-2}{n^3}, \frac{n-3}{n^3} \text{ \&c.}$$

whence the values of the first, second, third, &c. payment of the reversion will be

$$\frac{n-1}{n^3} - \frac{n-1}{n^4}, \frac{n-2}{n^3} - \frac{n-2}{n^4}, \frac{n-3}{n^3} - \frac{n-3}{n^4} \text{ \&c.}$$

And therefore, the value of the required reversion may be found by subtracting the value of the four equal joint lives, from the value of the three equal joint lives.

Now retaining the symbols used in corol. to quest. 75.

$$N^{\text{III}} = P - \frac{3}{n} Q + \frac{3}{nn} R - \frac{1-p}{n^3} S,$$

$$N^{\text{IV}} = P - \frac{4}{n} Q + \frac{6}{nn} R - \frac{4}{n^3} S + \frac{1-p}{n^4} T;$$

And

And therefore the value of the reversion required will be

$$\frac{1}{n} Q - \frac{2}{nn} R + \frac{3+p}{n^2} S - \frac{1-p}{n^3} T.$$

C O R O L.

Retaining the same symbols, and writing \mathcal{R} for the reversion of one life after one; \mathcal{R}^{ii} for the reversion of two equal joint lives after one; \mathcal{R}^{iii} for the reversion of three equal joint lives after one, &c.

Since $\mathcal{R} = \frac{1+p}{n} Q - \frac{1-p}{nn} R$, By quest. 94

$$\mathcal{R}^{\text{ii}} = \frac{1}{n} Q - \frac{2+p}{nn} R + \frac{1-p}{n^3} S, \quad 95$$

$$\mathcal{R}^{\text{iii}} = \frac{1}{n} Q - \frac{3}{nn} R + \frac{3+p}{n^3} S - \frac{1-p}{n^4} T; \quad 96$$

Therefore $\left\{ \begin{array}{l} \mathcal{R}^{\text{iv}} = \frac{1}{n} Q - \frac{4}{nn} R + \frac{6}{n^3} S - \frac{4+p}{n^4} T + \frac{1-p}{n^5} U, \\ \mathcal{R}^{\text{v}} = \frac{1}{n} Q - \frac{5}{nn} R + \frac{10}{n^3} S - \frac{10}{n^4} T + \frac{5+p}{n^5} U - \left(\frac{1-p}{n^6} \right) W; \end{array} \right.$

And the value of the reversion of m equal joint lives, after the failure of one life of the same age, will be

$$\frac{1}{n} Q - \frac{m}{n^2} R + \frac{m \cdot m - 1}{n^2 \cdot 2n} S - \frac{m \cdot m - 1 \cdot m - 2}{n^2 \cdot 2n \cdot 3n} T (m-1);$$

$$\left(\pm \frac{m+p}{n^m} Y + \frac{1-p}{n^{m+1}} Z \right).$$

In which expression Y and Z denote the $m+1$ th and $m+2$ th factors in the series above quoted $P, Q, R, \&c.$ and the signs throughout the whole will be alternately $+$ and $-$.

QUESTION XCVII.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, on the the longest of two lives of given ages, after the decease of the present possessor, whose age is also given.

SOLUTION.

Let the complement of the life of the present possessor be denoted by m ; and those of the expectants by t and n .

Then the probability of the possessor's dying in the first, second, third, &c. years, will be

$$1 - \frac{m-1}{m}, 1 - \frac{m-2}{m}, 1 - \frac{m-3}{m} \text{ \&c.}$$

And by quest. 76, the probability of one (at least) of the expectants surviving the first, second, third, &c. years will be

$$\begin{aligned} \frac{m-1}{m} + \frac{t-1}{t} &= \frac{m-1 \times t-1}{mt}, \\ \frac{m-2}{m} + \frac{t-2}{t} &= \frac{m-2 \times t-2}{mt}, \\ \frac{m-3}{m} + \frac{t-3}{t} &= \frac{m-3 \times t-3}{mt} \text{ \&c.} \end{aligned}$$

which probabilities, being severally multiplied into the corresponding probabilities of the possessor's dying, will give the values of the first, second, third, &c. payments of the reversion, viz.

$$\begin{array}{ccccc}
\frac{m-1}{m} + \frac{t-1}{t} - \frac{m-1 \times t-1}{mt} - \frac{n-1 \times m-1}{nm} - \frac{n-1 \times t-1}{nt} \\
\frac{m-2}{m} + \frac{t-2}{t} - \frac{m-2 \times t-2}{mt} - \frac{n-2 \times m-2}{nm} - \frac{n-2 \times t-2}{nt} \\
\frac{m-3}{m} + \frac{t-3}{t} - \frac{m-3 \times t-3}{mt} - \frac{n-3 \times m-3}{nm} - \frac{n-3 \times t-3}{nt} \\
\text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.}
\end{array}$$

$$\begin{aligned}
& \left(+ \frac{n-1 \times m-1 \times t-1}{nmt} \right), \\
& \left(+ \frac{n-2 \times m-2 \times t-2}{nmt} \right), \\
& \left(+ \frac{n-3 \times m-3 \times t-3}{nmt} \right), \\
& \text{&c.}
\end{aligned}$$

The present value of which expression, being compared with the first expression (given in quest. 80) for the value of the longest of three lives, will appear to differ therefrom, by

$$\frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} \text{ &c. the value of the life of the possessor.}$$

Therefore : if, from the value of the longest of the three lives of the possessor and two expectants, be taken the value of the possessor's life, the remainder will be the value of the reversion, of the longest of two lives after one life.

The application of which rule to the foregoing solutions will admit of eight cases, as in quest. 95.

CASE I. When the possessor is older than either of the expectants, their ages being unequal.

By the solution of quest. 80, the value of the longest of three lives is

P +

$$P + \frac{1}{tr^t} + \frac{1}{tr^m} + \frac{n}{mtr^n} \times Q + \frac{2}{mtr^n} + \frac{1}{mt^n} \times R - \frac{1-p}{nmt}$$

And (by quest. 56) the value of the possessor's life will be

$$P + \frac{1}{m^n} - \frac{1}{n} \times Q; \text{ which, subtracted from the former, leaves }$$

$$\frac{1}{tr^t} + \frac{1}{tr^m} + \frac{n}{mtr^n} + \frac{1}{n} - \frac{1}{m^n} \times Q$$

$\left(+ \frac{2}{mtr^n} + \frac{1}{mtr^n} \times R - \frac{1-p}{nmt} \right) S$, the value of the reversion, which may be deduced to

$$\frac{1}{r^t} + \frac{1}{r^m} \times nm + mt - mt - nn \times \frac{1}{rn} \times \frac{Q}{nmt} + \frac{2}{r^n} + \frac{1}{r^n} \left(\times \frac{nR}{nmt} - \frac{1-p}{nmt} \right) S$$

Nothing would be saved (in the numerical process) by subtracting the symbol of the possessor's life from the approximation to the value of the longest of the three lives found in quest. 81.

Therefore from the approximate value of the }
longest of the three lives - - - - - } 15,141,

Take the value of the possessor's life 7,333;

The remainder will be the value of the reversion 7,808.

It remains to compare these with the true result, *viz.*

From the true value of the longest of the three }
lives found in quest. 80 - - - - - } 15,190

Take the value of the possessor's life 7,333

Remains the true value of the reversion 7,857

It appears from the solution of the above case, that, where the value of the possessor's life is given (as in the tables annexed) this numerical process will be soon performed;

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formed ; and therefore the next 6 cases are here omitted, and we shall proceed to the 8th.

CASE 8. When the possessor and expectants are of equal ages.

This may be solved from case 1, by making $\frac{1}{x} =$

$\frac{1}{x} = \frac{1}{x} = p$; and ~~making~~; thus

$$\frac{1+2p}{x} Q + \frac{3p}{xx} R - \frac{1-p}{x^2} S.$$

Also, if from the approximation to the value of the longest of three equal lives, viz. $N + \frac{3x+1 \times x+1}{4xx} \times$
 $N - \frac{x-1}{6x}$; the value of the possessor's life (N) be taken,

the remainder $\frac{3x+1 \times x+1}{4xx} \times N - \frac{x-1}{6x}$ will be the value of the reversion.

The giving of a rule, in words, for which, will be unnecessary ; because it will differ from that in quest. 84 and 85, only, in not adding the value of the single life, to the quotient, found as therein directed.

QUESTION XCVIII.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, on the longest of three equal lives, after the decease of a fourth person of the same age.

SOLU.

SOLUTION.

Let the complement of their lives be denoted by n ;

Then the probabilities of the possessor's dying in the first, second, third, &c. year, will be

$$1 - \frac{n-1}{n}, 1 - \frac{n-2}{n}, 1 - \frac{n-3}{n} \text{ \&c.}$$

And by arguing, as in quest. 86, the probabilities of one, at least, of the expectants surviving the first, second, third, &c. year, will be

$$3 \times \frac{n-1}{n} - 3 \times \frac{n-1^2}{nn} + \frac{n-1^3}{n^3},$$

$$3 \times \frac{n-2}{n} - 3 \times \frac{n-2^2}{nn} + \frac{n-2^3}{n^3},$$

$$3 \times \frac{n-3}{n} - 3 \times \frac{n-3^2}{nn} + \frac{n-3^3}{n^3} \text{ \&c.}$$

which probabilities, being severally multiplied into the corresponding probabilities of the possessor's dying, will give the first, second, third, &c. payment.

$$3 \times \frac{n-1}{n} - 6 \times \frac{n-1^2}{nn} + 4 \times \frac{n-1^3}{n^3} - \frac{n-1^4}{n^4},$$

$$3 \times \frac{n-2}{n} - 6 \times \frac{n-2^2}{nn} + 4 \times \frac{n-2^3}{n^3} - \frac{n-2^4}{n^4},$$

$$3 \times \frac{n-3}{n} - 6 \times \frac{n-3^2}{nn} + 4 \times \frac{n-3^3}{n^3} - \frac{n-3^4}{n^4} \text{ \&c.}$$

which differ from the like expressions found for the value of the longest of four equal lives by the value of the single

$$\text{life } \frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} \text{ \&c.}$$

And therefore the value of the required reversion will be found by subtracting the value of the single life from the value of the longest of the four equal lives.

Now,

Now, if we use the same symbols as in the corol. to quest. 86,

$$L.N^{iv} = P + \frac{4p}{n} Q + \frac{6p}{nn} R + \frac{4p}{n^3} S - \frac{1-p}{n^4} T,$$

$$N = P - \frac{1-p}{n} Q;$$

Therefore the value of the reversion of an annuity, for the longest of three equal lives, after one, will be

$$\frac{1+3p}{n} Q + \frac{6p}{nn} R + \frac{4p}{n^3} S - \frac{1-p}{n^4} T.$$

COROL.

If $L.R^{ii}$ denote the reversion of the longest of two equal lives after one; $L.R^{iii}$ the reversion of the longest of three equal lives after one, &c. then

$$\text{since } R = \frac{1+p}{n} Q - \frac{1-p}{nn} R, \quad \text{quest. 94}$$

$$L.R^{ii} = \frac{1+2p}{n} Q + \frac{3p}{nn} R - \frac{1-p}{n^3} S, \quad 97$$

$$L.R^{iii} = \frac{1+3p}{n} Q + \frac{6p}{nn} R + \frac{4p}{n^3} S - \frac{1-p}{n^4} T;$$

$$\text{then } L.R^{iv} = \frac{1+4p}{n} Q + \frac{10p}{nn} R + \frac{10p}{n^3} S + \frac{5p}{n^4} T -$$

$$\left(\frac{1-p}{n^5} V \right);$$

And the value of the reversion of the longest of m equal lives, after one life of equal age with the former, will be.

$$\frac{1+mp}{n} Q + \frac{m+1 \cdot m}{n \cdot 2n} R + \frac{m+1 \cdot m \cdot m-1}{n \cdot 2n \cdot 3n} S \quad (m)$$

$$\left(- \frac{1-p}{n^{m+1}} Z \right)$$

Where

Where Z is the $m+2$ th factor of the series $P, Q, R, \&c.$ mentioned in quest. 20; and the term into which it is to be multiplied, will always be negative.

QUESTION XCIX.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the joint lives of two persons, whose ages are also given.

SOLUTION.

Let the complements of the two lives in possession be m and n ; and that of the expectant t .

Now since the probability of both of the possessors living, to the end of the first, second, third, &c. year, is

$$\frac{m-1 \times n-1}{mn}, \frac{m-2 \times n-2}{mn}, \frac{m-3 \times n-3}{mn} \&c.$$

Therefore the probability, that one of them, at least, will fail, in the first, second, third, &c. year, will be

$$1 - \frac{m-1 \times n-1}{mn}, 1 - \frac{m-2 \times n-2}{mn}; 1 - \frac{m-3 \times n-3}{mn} \&c.$$

And the probability of the continuance of the expectant's life, for one, two, three, &c. years, will be

$$\frac{t-1}{t}, \frac{t-2}{t}, \frac{t-3}{t} \&c.$$

which probabilities, being severally multiplied by the corresponding probabilities of one of the possessors failing;

will give the value of the first, second, &c. payments, of the reversion, viz.

$$\frac{t-1}{t} - \frac{m-1 \times n-1 \times t-1}{mnt}, \frac{t-2}{t} - \frac{m-2 \times n-2 \times t-2}{mnt}, \&c.$$

the present value, of which payments, will be

$$\frac{t-1}{tr} - \frac{m-1 \times n-1 \times t-1}{mnt r} + \frac{t-2}{tr^2} - \frac{m-2 \times n-2 \times t-2}{mnt r^2} \&c.$$

where $\frac{t-1}{tr} + \frac{t-2}{tr^2} + \frac{t-3}{tr^3} \&c.$ is the value of the expectants life.

$$\text{and } \frac{m-1 \times n-1 \times t-1}{mnt r} + \frac{m-2 \times n-2 \times t-2}{mnt r^2} \&c. \text{ is}$$

the value of the three joint lives, of the two possessors and the expectant.

Therefore; if, from the value of the expectant's life, the value of the three joint lives (of the two possessors and the expectant) be taken; the remainder will be the value of one life, after two joint lives.

New here (as in quest. 97) it will appear, that, when a table of the values of single lives is at hand, nothing will be saved, in the numerical solution, by finding the difference of those two values, as expressed in symbols: it is therefore here thought sufficient to give an example, and answer it by the former results.

What is the value of the reversion of an annuity, to continue during the life of a person aged 43 years, after the joint lives of two persons of the respective ages of 54 and 66; allowing compound interest at four per Cent?

First, by the true method,

From the value of the expectant's life (quest. 56) 12,683,
Take the value of the three joint lives (quest. 69) 5,152,

Remains the value of the reversion

7,531.
Secondly,

REPOSITORY.

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Secondly, by the approximation,	
From the value of the expectant's life	12,683,
Take the value of the three joint lives	5,199,
	<hr/>
Remains the value of the reversion	7,484.

C O R O L.

If the given lives are equal, and the same symbols be used as in the corol. to quest 75 ; then

$$N = P - \frac{1-p}{n} Q,$$

$$N^{III} = P - \frac{3}{n} Q + \frac{3}{nn} R - \frac{1-p}{n^3} S;$$

therefore the value of the required reversion will be

$$\frac{2+p}{n} Q - \frac{3}{nn} R + \frac{1-p}{n^3} S.$$

QUESTION C.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the joint lives of three other persons, each of the same age.

SOLUTION.

Let the complement of those lives be n .

Then the probability that one, at least, of the three possessors will die in the first, second, third, &c. year, will be

$$1 - \frac{n-1}{n^3}, 1 - \frac{n-2}{n^3}, 1 - \frac{n-3}{n^3} \text{ \&c.}$$

P 2

which

which, being severally multiplied by the respective probabilities of the expectant's living to the end of the first, second, third, &c. year, will give the first, second, third, &c. payment of the reversion, viz.

$$\frac{n-1}{n} - \frac{n-1^4}{n^4}, \frac{n-2}{n} - \frac{n-2^4}{n^4}; \frac{n-3}{n} - \frac{n-3^4}{n^4}.$$

Therefore; if, from the value of the single life, the value of the four joint lives be taken, the remainder will be the value of the reversion.

Now, if the same symbols be used as in corol. to quest. 75, then

$$N = P - \frac{1-p}{n} Q;$$

$$N^iv = P - \frac{4}{n} Q - \frac{6}{nn} R - \frac{4}{n^3} S + \frac{1-p}{n^4} T;$$

therefore $\frac{3+p}{n} Q - \frac{6}{nn} R + \frac{4}{n^3} S - \frac{1-p}{n^4} T$ will be the value of the reversion required.

C O R O L.

If ^{ii}R denote the value of the reversion of one life, after two equal joint lives; the ^{iii}R the reversion of one life, after three such lives &c. then

$$\text{since } R = \frac{1+p}{n} Q - \frac{1-p}{nn} R, \quad \text{quest. 94}$$

$$^{ii}R = \frac{2+p}{n} Q - \frac{3}{nn} R + \frac{1-p}{n^3} S, \quad 99$$

$$^{iii}R = \frac{3+p}{n} Q - \frac{6}{nn} R + \frac{4}{n^3} S - \frac{1-p}{n^4} T;$$

$$^{iv}R = \frac{4+p}{n} Q - \frac{10}{nn} R - \frac{10}{n^3} S - \frac{5}{n^4} T + \frac{1-p}{n^5} V.$$

And

And the value of the reversion of one life, after m joint lives of equal ages, will be

$$\frac{m+1}{n} Q - \frac{m+1 \cdot m}{n \cdot 2n} R + \frac{m+1 \cdot m \cdot m-1}{n \cdot 2n \cdot 3n} S (m) + \frac{1-p}{n \cdot m+1} Z;$$

where Z is the $m+1$ th factor of the series P , Q , R , &c. whose values are given in quest. 20, and the terms will be alternately $+$ & $-$, throughout the whole expression.

QUESTION CI.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the longest liver of two persons, whose ages are also given.

SOLUTION.

Let the complements of the lives in possession be m and n , and that of the expectant t .

Now the probabilities of both of the possessors dying in the first, second, third, &c. years, will (*per* corol. quest. 29) be

$$\begin{aligned} \left(1 - \frac{n-1}{n} \times 1 - \frac{m-1}{m}\right) &= 1 - \frac{n-1}{n} - \frac{m-1}{m} + \frac{n-1 \times m-1}{nm} \\ \left(1 - \frac{n-2}{n} \times 1 - \frac{m-2}{m}\right) &= 1 - \frac{n-2}{n} - \frac{m-2}{m} + \frac{n-2 \times m-2}{nm} \\ \left(1 - \frac{n-3}{n} \times 1 - \frac{m-3}{m}\right) &= 1 - \frac{n-3}{n} - \frac{m-3}{m} + \frac{n-3 \times m-3}{nm} \\ &\text{&c.} \qquad \qquad \qquad \text{&c.} \end{aligned}$$

and the probability of the expectant's living to the end of the first, second, third, &c. years, is

$\frac{t-1}{t}$, $\frac{t-2}{t}$, $\frac{t-3}{t}$ &c. which probabilities, being severally multiplied into the corresponding probabilities of the two possessors dying, and the present values of the products being taken, will give the value of the reversion, viz.

$$\begin{aligned} & \frac{t-1}{tr} - \frac{n-1 \times t-1}{ntr} - \frac{m-1 \times t-1}{mtr} + \frac{n-1 \times m-1 \times t-1}{nmtr}, \\ & + \frac{t-2}{tr^2} - \frac{n-2 \times t-2}{ntr^2} - \frac{m-2 \times t-2}{mtr^2} + \frac{n-2 \times m-2 \times t-2}{nmtr^2}, \\ & + \frac{t-3}{tr^3} - \frac{n-3 \times t-3}{ntr^3} - \frac{m-3 \times t-3}{mtr^3} + \frac{n-3 \times m-3 \times t-3}{nmtr^3}, \\ & \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \end{aligned}$$

Now if the two first expressions, given in questions 76 and 80 for the respective values of the longest of two and three lives, be compared with this; it will appear to be their difference: and, therefore

If, from the value of an annuity on the longest of the three lives of the two possessors and expectant, the value of an annuity on the longest of the two possessors lives be taken; the remainder will be the value of the reversion, of one life, after the longest of two lives.

The application of this, to the former solutions, will likewise admit of eight cases.

CASE I. When the possessors are of unequal ages, and both elder than the expectant.

Then, the value of the annuity on the longest of the three lives, will (*per* quest. 80) be

$$P + \frac{1}{tr^t} + \frac{1}{tr^m} + \frac{n}{mtr^n} \times 2 + \frac{2}{mtr^n} + \frac{1}{mtr^n} \times R - \left(\frac{1-p}{nmt} \right)$$

and the value of the annuity, on the longest of the two possessors lives, will be

$$P + \frac{1}{mr^m} + \frac{1}{mtr^n} \times 2 - \frac{1}{nm} - \frac{1}{nmtr^n} \times R$$

the difference of which two expressions will be

$$\frac{1}{tr^t} - \frac{1}{m} - \frac{1}{t} \times \frac{1}{r^m} - \frac{1}{m} - \frac{n}{mt} \times \frac{1}{r^n} \times 2 + \left(\frac{1}{nm} + \frac{1}{mtr^n} + \frac{2}{mt} - \frac{1}{nm} \times \frac{1}{r^n} \right) \times R - \frac{1-p}{nmt} S;$$

$$\text{Or } \frac{nm}{rt} - \frac{t-m \times n}{r^n} - \frac{t-n \times n}{r^n} \times \frac{2}{nmt} + \left(t + \frac{n}{r^m} + \frac{2n-t}{r^n} \times \frac{R}{nmt} - \frac{1-p}{nmt} \right) S$$

the value of the reversion required.

EXAMPLE.

A, who is 43 years of age, is entitled to the reversion of an estate of one pound *per annum* for his life, after the decease of his father (who is 66 years of age) and of his mother-in-law (aged 54) who is jointured thereon; I demand the value of *A*'s interest in the estate, allowing four *per Cent*.

By the first method,

From the value of the longest of the three given	}	15,190,
lives, found <i>per quest</i> . 80 - - - -		
Subtract the value of the longest of the two	}	11,904;
possessors lives, found <i>per quest</i> . 76 - -		

Remains the value of the reversion required 3,286.

By the method last given;

P 4

Here

From the complement of the younger possessor's life, subtract one; and divide the remainder by six times the rate, or find this quotient in the last table.

Subtract the quotient from the value of the younger possessor's life; and multiply the remainder by his complement more one, reserving the product.

From the complement of the elder possessor's life, subtract one; and divide the remainder by six times the rate, or find this quotient in table the last; subtract the quotient from the value of the elder possessor's life, reserving the remainder.

From twice the expectant's complement, subtract the elder possessor's complement, and one; multiply the remainder by the elder possessor's complement more one; and this product by the remainder above reserved; divide this last product, by twice the younger possessor's complement.

From the product, above reserved, subtract the last found quotient, and divide the remainder by twice the expectant's complement.

To the difference of the values of the single lives, of the expectant and younger possessor, add the quotient last found, and their sum will be the value of the reversion required.

E X A M P L E.

If the possessor's be severally of the ages of 54 and 66 years, the expectant 43, and interest four per Cent. as above.

Then 12,683 will be the value of the expectant's life,

10,478

younger possessor's life,

7,333

elder possessor's;

And $(86 - 43 =) 43$ will be the expectant's complement

$(86 - 54 =) 32$

younger possessor's,

$(86 - 66 =) 20$

elder possessor's;

Also $(1\frac{1}{4}, 04 =) 1,04$

the rate.

Now

Now $(32-1=) 31$ being divided by $(6 \times 1,04=) 6,24$ will quote 4,9689, which, being taken from 10,478, the remainder, will be 5,510; which, being multiplied by $(32+1=) 33$ will give 181,83, for the product to be reserved.

Also $(20-1=) 19$ being divided by $(6 \times 1,04=) 6,24$, will quote 3,045, which, subtracted from 7,333, will give 4,288, for the remainder to be reserved.

From $(2 \times 43=) 86$, subtract 20 & 1, the remainder will be 65; which, being multiplied by $(20+1=) 21$, will produce 1365, and this product, being multiplied by 4,288 (the remainder above reserved) produces 5853,12; which, being divided by $(2 \times 32=) 64$, will quote 91,455.

If the last found quotient 91,455, be subtracted from 181,83 (the product above reserved) and the remainder, 90,375, be divided by $(2 \times 43=) 86$, the quotient will be 1,051.

Lastly; if, to $(12,683-10,478=) 2,205$, the above quotient 1,051, be added, their sum, 3,256, will be the value of the reversion required.

CASE 2. When the possessors are of equal ages, and both elder than the expectant.

The solution of this case may be deduc'd from the solution of case 1, by writing n for m ; and $\frac{1}{ra}$ for $\frac{1}{r^m}$; as follows,

$$\frac{m}{r^t} - \frac{1-n \times 2}{r^n} \times \frac{2}{nt} + t + \frac{3n-t}{r^n} \times \frac{2}{nnt} - \frac{1-p}{nnt} S$$

An approximation to this value may, also, be deduc'd from the approximation given in case 1, by writing n for m ; and N for M , viz.

$$F = N + \frac{1}{2t} \times \overline{n+1} \times N - \frac{n-1}{6r} - \overline{n+1} \times \frac{2t-n-1}{2n} \times \left(N - \frac{n-1}{6r} \right)$$

$$\text{Or, } F = N + \frac{1}{2t} \times \overline{n+1} \times N - \frac{n-1}{6r} \times 1 - \frac{2t-n-1}{2n} ;$$

$$\text{But } 1 = \frac{2t-n-1}{2n} = \left(\frac{2n-2t+n+1}{2n} = \right) \frac{3n+1-2t}{2n} ;$$

Therefore the value of the reversion will become

$$F = N + N - \frac{n-1}{6r} \times \frac{\overline{n+1} \times \overline{3n+1-2t}}{4nt}$$

Which, expressed in words at length, follows :

The rule, for finding the present value of the reversion of an annuity, to continue during a life, of a given age, after the decease of two other persons of equal ages, and both elder than the former; having the value of the single lives given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which each person wants of 86, be called their complements of life, and let one pound, and its interest for one year, be called the rate.

From the complement of the life of one of the possessors, subtract one, and divide the remainder by six times the rate, or find the quotient in the last table; subtract the quotient from the value of one of the possessors life, reserving the remainder.

Find the difference between, a number greater by one than three times one of the possessor's complement, and twice the expectant's complement; multiply this difference by the possessor's complement more one; and multiply this product by the remainder, above reserved; divide this last product by six times the product of the two different complements.

Then

Then, if the number, which is greater by one than the possessor's complement exceeds twice the expectant's complement, add this quotient to, or if that number be less than twice the expectant's complement, subtract this quotient from, the difference of the values of the single lives of the expectant and one of the possessors, so shall their sum, or remainder, be the value of the reversion required.

E X A M P L E.

If the expectant be 43 years old, the two possessors each 66, and interest at four *per Cent*.

Then 12,683 will be the value of the life of the expectant;
7,333. each possessor;

And $(86 - 43 =) 43$ will be the expectant's complement.
 $(86 - 66 =) 20$ the possessor's;

Also $(1 + .04 =) 1,04$ the rate.

Now if $(20 - 1 =) 19$, be divided by $(6 \times 1,04 =) 6,24$, the quotient will be 3,045, which, subtracted from 7,333, will leave 4,288, for the remainder to be reserved.

The number which is greater by one than $(3 \times 20 =) 60$ will be 61; and $(2 \times 43 =) 86$; the difference of which is 25; which, being multiplied by $(20 + 1 =) 21$, will produce 525; and this product, being multiplied by 4,288, (the remainder above reserved) will produce 2251,2; which last product, being divided by $(4 \times 20 \times 43 =) 3440$, will quote 0,654.

Now (because 86 exceeded 61) the quotient 0,654, is to be subtracted from $(12,683 - 7,333 =) 5,350$; and the remainder, 4,696, will be the value of the reversion required.

CASE 3. When the expectant is elder than the possessors, and they are of unequal ages.

Let

REPOSIT

From the complement of the expectation and divide the remainder by six times the quotient by the last table; subtract the value of the expectant's life; and multiply the product that sum by itself; and multiply the remainder above reserved.
 Divide the two possessors complements; so shall the value of the reversion required.

EXAMPLE.

If the possessors are severally 43 and 54 years of age, and interest at four per Cent.
 Then the value of the expectation will be the value of the expectant's life; and the complements thereof will be the complements of the possessors;
 And the rate; being divided by (6x) the rate, which, being subtracted from the value of the reversion, will be the value of the reversion required.

multiplied by
 also multiplied by
 will produce
 by (4x)
 value

AT
 IN

Let n be the complement of the expectant's life, and t and m those of the possessor's.

Then will the value of an annuity on the longest of the two possessor's lives be

$$P + \frac{1}{t^r} + \frac{1}{t^m} \times Q - \frac{1}{t^r m} - \frac{1}{t m^r} \times R;$$

which, being taken from the value of the longest of the three lives, will leave

$$\frac{n}{m t^r} \times Q + \frac{1}{t^m} + \frac{2}{m t^r} \times R - \frac{1-p}{m t} S,$$

$$\text{Or, } \frac{1}{m t} \times n p Q + 1 + 2 p \times R - \frac{1-p}{m} S.$$

Now the approximation, to the longest of the possessor's lives, will be

$F + M - \frac{m-1}{6r} \times \frac{m+1}{2t}$; which, being taken from the approximation, to the value of the longest of the three lives, there will remain,

$N - \frac{n-1}{6r} \times \frac{n+1}{4mt}$, for the value of the reversion required.

Which, expressed in words at length, follows :

The rule for finding the present value of the reversion of an annuity, to continue during the life of a person of a given age, after the decease of two persons of given ages, both younger than the former, having the value of the expectant's life given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let the number of years, which each person wants of 86, be called their complements of life; and let one pound, and its interest for one year, be called the rate.

From

From the complement of the expectant's life, subtract one ; and divide the remainder by six times the rate, or find the quotient by the last table ; subtract the quotient from the value of the expectant's life, reserving the remainder.

To the complement of the expectant's life, add one ; multiply that sum by itself ; and multiply the product by the remainder above reserved.

Divide the product last found, by four times the product of the two possessors complements ; so shall the quotient be the value of the reversion required.

E X A M P L E.

If the possessors are severally 43 and 54 years old, the expectant 66, and interest at four *per Cent*.

Then 7,333 will be the value of the expectant's life,

(86—66=) 20 the complement thereof,

(86—54=) 32 } the complements of the two

(86—43=) 43 } possessors ;

And (1+04=) 1,04, the rate ;

Now if (20—1=) 19 be divided by (6×1,04=) 6,24, the quotient will be 3,045, which, being subtracted from 7,333, will leave 4,288, for the remainder to be reserved.

And if (20+1=) 21, be multiplied by 21, the product will be 441 ; which, being also multiplied by 4,288 (the remainder above reserved) will produce 1891,008.

Also, if 1891,008 be divided by (4×43×32=) 5504, the quotient 0,343, will be the value of the annuity required.

CASE 4. When the possessors are of equal ages, and both younger than the expectant.

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This case may be performed by the rules given for the last case; the whole difference being in the last divisor; which, when the possessors ages are equal, will, in the first rule, be mm , instead of mt ; and in the other, $4mm$, instead of $4mt$.

CASE 5. When the possessors are one elder, and the other younger, than the expectant.

Let m be the complement of the expectant's life, and s and n those of the possessors.

Then the value of the annuity on the longest of the two possessors lives will be

$$P + \frac{1}{tr^t} + \frac{1}{sn} \times 2 - \frac{1}{nt} - \frac{1}{ntsn} \times R;$$

which, being subtracted from the value of an annuity, on the longest of the three lives, there will remain

$$\frac{1}{tr^m} - \frac{1}{t} - \frac{n}{mt} \times \frac{1}{sn} \times 2 +$$

$$\left(\frac{1}{nt} + \frac{1}{mtsn} + \frac{2}{mt} - \frac{1}{nt} \times \frac{1}{sn} \right) \times R - \frac{1-p}{mnt} S, \text{ Or } \\ \frac{m}{r^m} - \frac{2m-n}{r^n} \times \frac{2}{mt} + m + \frac{n}{r^m} + \frac{2n-m}{r^n} \times \frac{R}{mnt} - \left(\frac{1-p \times S}{mnt} \right)$$

the value of the reversion required.

Now the approximation to the value of an annuity for the longest of the two lives of the possessors will be

$$E - N - \frac{n-1}{6r} \times \frac{n+1}{2t}; \text{ which, being taken from the value of the longest of the three lives, will leave.}$$

$$N - \frac{n-1}{6r} \times \frac{n+1}{4mt} - N - \frac{n-1}{6r} \times \frac{n+1}{2t} + M - \frac{n-1}{2m} \times \left(\frac{n+1}{2t} \right)$$

$$\text{Now } \frac{n+1}{4mt} = \frac{n+1}{2t} \times \frac{n+1}{2m}; \text{ and } \frac{n+1}{2m} - 1 = \left(-\frac{2m-n-1}{2m} \right)$$

whence the value of the reversion will be

$$M - \frac{n-1}{6r} \times \frac{n+1}{2t} - N - \frac{n-1}{6r} \times \frac{n+1 \times 2m - n - 1}{4mt}; \text{ or}$$

$$\frac{1}{2t} \times M - \frac{n-1}{6r} \times \frac{n+1}{2t} - N - \frac{n-1}{6r} \times \frac{n+1 \times 2m - n - 1}{2m}$$

Which, in words at length, is as follows :

The rule to find the present value of the reversion of an annuity, to continue during the life of a person of a given age, after the decease of two other persons, one of which is elder, and the other younger, than the expectant; having the values of the single lives of the expectant, and eldest possessor, given; allowing compound interest at a given rate; and supposing the decrements of life to be equal.

Let the number of years, which each of the persons want of 86, be called their complements of life; and let one pound, and its interest for one year, be called the rate.

From the complement of the expectant's life, subtract one; and divide the remainder by 6x times the rate, or find the quotient in the last table.

Subtract this quotient from the value of the expectant's life; and multiply the remainder by the expectant's complement more one, reserving the product.

From

From the elder possessor's complement subtract one, and divide the remainder by six times the rate, or find this quotient in the last table; subtract the quotient from the value of the elder possessor's life, reserving the remainder.

From twice the expectant's complement, subtract the complement of the elder possessor, and one; multiply the remainder by the elder possessor's complement more one; and multiply that product by the remainder above reserved; dividing the last product by twice the expectant's complement.

From the product above reserved, take the last found quotient; and divide the remainder by twice the complement of the younger possessor, so shall the quotient be the value of the reversion required.

EXAMPLE

If the expectant be 54 years old, the possessors severally 43, and 66; and interest at four per Cent.

Then 10,478 will be the value of the expectant's life,

7,333,

that of the elder possessor

(86—66=) 20 the complement of the elder possessor

(86—54=) 32

expectant,

(86—43=) 43

younger possessor;

And (1+0,04=) 1,04 the rate.

Now if (32—1=) 31 be divided by (6×1,04=) 6,24, the quotient will be 4,968; which, being subtracted from 10,478; and the remainder 5,510, be multiplied by (32+1=) 33, the product will be 181,83, which is to be reserved.

And if (20—1=) 19, be divided by 6,24; and the quotient 3,045, be subtracted from 7,333, the remainder 4,288, is to be reserved.

Again

Again $(2 \times 32 - 20 - 1 =) 43$, being multiplied by $(20 + 1 =) 21$, the product will be 903; which, being multiplied by 4,288 (the remainder above reserved) will produce 3872,0; and this divided by $(2 \times 32 =) 64$, will quote 60,5.

Lastly, if from 181,83 (the product above reserved) this quotient, 60,5, be subtracted; and the remainder 121,33, be divided by $2 \times 43 =) 86$, the quotient 1,411, will be the value of the reversion required.

CASE 6. When the elder possessor and expectant are of the same age.

The solution of this case may be derived from the last; by writing n for m ; and r^n for r^m ; whence the value of the reversion will become

$$\frac{2}{tr^n} + n + \frac{2n}{r^n} + \frac{R}{nnt} - \frac{1-p}{nnt} S, \text{ Or,}$$

$$\frac{2p}{t} + 1 + 2p \times \frac{R}{nt} - \frac{1-p}{nnt} S;$$

And the approximation thereto will, by writing n for m , and N for M , become

$$\frac{1}{2t} \times N - \frac{n-1}{6r} \times n + 1 - N - \frac{n-1}{6r} \times \frac{n+1 \times 2n-n-1}{2n}$$

$$\text{that is } \frac{1}{2t} \times N - \frac{n-1}{6r} \times n + 1 \times 1 - \frac{n-1}{2n}$$

But $1 - \frac{n-1}{2n} = \left(\frac{2n-n+1}{2n} \right) = \frac{n+1}{2n}$; therefore the approximation will become

$$\frac{1}{2t} \times N - \frac{n-1}{6r} \times \frac{n+1}{2n}, \text{ or } N - \frac{n-1}{6r} \times \frac{n+1}{4nt}; \text{ which}$$

may be expressed by the words given in the rule to case 3.

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CASE 7. When the younger possessor and expectant are of the same age.

This may be also deduced from case the 5th, by writing m for t ; when the value of the reversion will be

$$\frac{m}{r^m} - \frac{m-n}{r^n} \times \frac{2}{mm} + \frac{n}{r^m} + \frac{2n-m}{r^n} \times \frac{R}{mm} - \frac{1-p}{mm} S;$$

and the approximation thereto

$$\frac{1}{2m} \times M - \frac{m-1}{6r} \times \frac{m+1}{1-N} - \frac{n-1}{6r} \times \frac{n+1 \times 2m-n-1}{2m};$$

which may be expressed, in the same words, as the rule in case 5.

CASE 8. When the expectant and the possessors are of the same age.

The solution of this case may be deduced from any of the former (suppose the 6th) by writing n for t , whence the value of the reversion will become

$$\frac{2}{n} + \frac{1}{1+2p} \times \frac{R}{nn} - \frac{1-p}{nnn} S;$$

and the approximation thereto, $N - \frac{n-1}{6r} \times \frac{n+1}{4nn}$;

But $(n+1)^2 = nn+2n+1 = (nn+2n) + 1 = n \times n + 2$ nearly;

Therefore, for the approximation may be wrote,

$$N - \frac{n-1}{6r} \times \frac{n \times n + 2}{4nn}, \text{ or } N - \frac{n-1}{6r} \times \frac{n+2}{4n}.$$

Which, expressed in words at length, follows:

The rule for finding the value of the reversion of an annuity, which is to continue during a life of a given age, after the decease of two other persons of the same age; having the value of the single life given, allowing compound interest at a given rate, and supposing the decrements of life to be equal.

Let

Let the number of years, which the given age wants of 86, be called the complement of life; and let one pound, and its interest for one year, be called the rate.

From the complement of life, subtract one; and divide the remainder by six times the rate, or find this quotient in the last table.

Subtract this quotient from the value of the single life; multiply the remainder by the complement more two; and divide the product by four times the complement; so shall the quotient be the value of the reversion required.

EXAMPLE.

If the three persons be each 66 years of age, and the rate of interest four *per.Cent.*

Then 7,333 will be the value of the single life.

$(86 - 66 =) 20$, will be the complement of life.

And $(1 + .04 =) 1.04$, the rate.

If $(20 - 1 =) 19$, be divided by $(6 \times 1.04 =) 6.24$, the quotient will be 3.045; which, being subtracted from 7,333, leaves 4,288; this being multiplied by $(20 + 2 =) 22$, produces 94,34; which, divided by $(4 \times 20 =) 80$, gives 1,179, for the value of the reversion required.

QUESTION CII.

Supposing the decrements of life to be equal, it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the longest liver of three persons of the same age.

SOLU-

SOLUTION.

Let the complement of life be represented by n ; then will the probability of the failing of the three possible lives, in the first, second, third, &c. years, be

$$1 - \frac{n-1}{n}, 1 - \frac{n-2}{n}, 1 - \frac{n-3}{n} \text{ &c.}$$

$$\begin{aligned} \text{Or } 1 - \frac{n-1 \times 3}{n} + \frac{n-1^2 \times 3}{n^2} - \frac{n-1^3}{n^3}, \\ 1 - \frac{n-2 \times 3}{n} + \frac{n-2^2 \times 3}{n^2} - \frac{n-2^3}{n^3}, \\ 1 - \frac{n-3 \times 3}{n} + \frac{n-3^2 \times 3}{n^2} - \frac{n-3^3}{n^3}, \text{ &c.} \end{aligned}$$

which being severally multiplied by $\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}$

&c. the probabilities of the expectant's living one, two, three, &c. years, and the present worths of those probabilities taken, will produce

$$\begin{aligned} \frac{n-1}{nr} - \frac{n-1^2 \times 3}{n^2 nr} + \frac{n-1^3 \times 3}{n^3 nr} - \frac{n-1^4}{n^4 r}, \\ \frac{n-2}{nr^2} - \frac{n-2^2 \times 3}{n^2 nr^2} + \frac{n-2^3 \times 3}{n^3 nr^2} - \frac{n-2^4}{n^4 r^2}, \\ \frac{n-3}{nr^3} - \frac{n-3^2 \times 3}{n^2 nr^3} + \frac{n-3^3 \times 3}{n^3 nr^3} - \frac{n-3^4}{n^4 r^3}, \\ \text{&c.} \quad \text{&c.} \quad \text{&c.} \quad \text{&c.} \end{aligned}$$

for the value of the reversion required.

Now

$$\begin{array}{l}
 \text{Now} \left\{ \begin{array}{l} \frac{n-1}{nr} + \frac{n-2}{nr^2} + \frac{n-3}{nr^3} \text{ \&c.} \\ \frac{n-1^2}{n^2r} + \frac{n-2^2}{n^2r^2} + \frac{n-3^2}{n^2r^3} \text{ \&c.} \\ \frac{n-1^3}{n^3r} + \frac{n-2^3}{n^3r^2} + \frac{n-3^3}{n^3r^3} \text{ \&c.} \\ \frac{n-1^4}{n^4r} + \frac{n-2^4}{n^4r^2} + \frac{n-3^4}{n^4r^3} \text{ \&c.} \end{array} \right\} \begin{array}{l} \text{is the value of the joint} \\ \text{lives of} \end{array} \left\{ \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right\} \begin{array}{l} \text{persons of the given age by} \\ 56 \\ 65 \\ 72 \\ 76 \end{array}
 \end{array}$$

Therefore the value of the reversion required will be, the single life, less thrice the value of two equal joint lives, more thrice the value of three equal joint lives, more the value of four equal joint lives;

That is per corol. to quest. 75.

$$P - \frac{1-p}{n} Q,$$

$$-3P + \frac{3 \times 2}{n} Q - \frac{3 \times 1-p}{nn} R,$$

$$+3P - \frac{3 \times 3}{n} Q + \frac{3 \times 3}{nn} R - \frac{3 \times 1-p}{n^3} S,$$

$$-P + \frac{4}{n} Q - \frac{6}{nn} R + \frac{4}{n^3} S - \frac{1-p}{n^4} T;$$

The sum of all which, viz.

$\frac{p}{n} Q + \frac{3p}{nn} R + \frac{1+3p}{n^3} S - \frac{1-p}{n^4} T,$ will be the value of the reversion required.

QUESTION

QUESTION CIII.

Supposing the decrements of life to be equal ; it is required to find the value of the reversion of an annuity, which is to continue during the life of a person of a given age, after the longest liver of four persons of the same age.

SOLUTION.

By reasoning in the same manner as in the last question, it will appear, that the value of the required reversion will be, the value of the single life, less four times the value of two equal joint lives, more six times the value of three equal joint lives, more four times the value of four equal joint lives, less the value of five equal joint lives.

That is, per corol. to quest. 75.

$$P - \frac{1-p}{n} Q,$$

$$- 4P + \frac{4 \times 2}{n} Q - \frac{4 \times 1-p}{nn} R,$$

$$+ 6P - \frac{6 \times 3}{n} Q + \frac{6 \times 3}{nn} R - \frac{6 \times 1-p}{n^3} S,$$

$$- 4P + \frac{4 \times 4}{n} Q - \frac{4 \times 6}{nn} R + \frac{4 \times 4}{n^3} S - \frac{4 \times 1-p}{n^4} T,$$

$$+ P - \frac{5}{n} Q + \frac{10}{nn} R - \frac{10}{n^3} S + \frac{5}{n^4} T - \frac{1-p}{n^5} V;$$

The sum of all which, viz.

$$\frac{p}{n} Q + \frac{4p}{nn} R + \frac{6p}{n^3} S + \frac{1+4p}{n^4} T - \frac{1-p}{n^5} V,$$

will be the value of the reversion required.

COROL.

C O R O L. I.

Hence, the value of the reversion of one life, after the longest of m equal lives, will be

$$\frac{p}{n} Q + \frac{mp}{nn} R + \frac{\overline{m \cdot m - 1}}{nn \cdot Zn} S + \frac{\overline{m \cdot m - 1 \cdot m - 2}}{nn \cdot 2n \cdot 3n} T (m-1) \\ \left(+ \frac{1 + mp}{n^{2m}} Y - \frac{1-p}{n^{m+1}} Z; \right.$$

Where Y , and Z , are the $\overline{m+1}$ th and $\overline{m+2}$ th terms in the series of factors, mentioned in quest. 20, and all the terms in the expression are affirmative, but the last.

C O R O L. II.

By comparing the above expression, with those given for the longest of any number of lives (in corol. to quest. 86) it will appear, that the reversion of one life after the longest of m equal lives, will be the difference between the longest of $\overline{m+1}$ equal lives, and the longest of m equal lives.

Q U E S T I O N C I V.

A table of observations, deduced from the bills of mortality of any place, being given; to find the complement of that life, whose value (being computed according to the hypothesis of equal decrements) shall be nearly equal to the value of the given life, computed from the given table of observations.

Let x be the required complement of life; then the probabilities of the life's continuing, to the extremity of old age, will (by arguing as before) be

$$\frac{n-1}{n} + \frac{n-2}{2} + \frac{n-3}{n} + \frac{n-4}{n} (n)$$

which being an arithmetical progression, whose greatest term is $\frac{n-1}{n}$, least term nothing, and number of terms n ;

the sum thereof, will be $\frac{n-1}{n} + 0 \times \frac{n}{2}$ (by quest. 7, part

$$\text{II. vol. I. Or } \left(\frac{n-1}{n} \times \frac{n}{2} = \right) \frac{n-1}{2}$$

This expression should (in order to make the lives of the same value) be equal to the sum of a like number of terms, taken from the given table of observations, *viz.* those which express the probabilities of the given life's continuing to the age of 86, supposed as before to be the extremity of old age; the numerators of which terms will, if the table of observations be disposed in the manner of those given in page 157 and 159, be found in the second column thereof, and may be added together upon the face of the table; their common denominator being the number, which, in the same column, stands against the given age.

Now if s represent the sum of those numerators, and a their common denominator, then the following equation will arise

$$\frac{n-1}{2} = \frac{s}{a};$$

$$\text{Whence } n-1 = \frac{2s}{a}, \text{ and } n = \frac{2s}{a} + 1.$$

Upon which principle the following table is composed, shewing those complements, which correspond to the several ages therein mentioned, according to the table of observations, deduced from the bills of mortality of London.

Age

Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.	Age	Comp.
8	71,5	20	57,8	32	45,5	44	36,2	56	27,3	68	18,3		
9	70,7	21	56,7	33	44,7	45	35,5	57	26,5	69	17,7		
10	69,8	22	55,6	34	43,8	46	34,8	58	25,8	70	17,2		
11	68,7	23	54,5	35	42,9	47	34,1	59	25,1	71	16,4		
12	67,6	24	53,3	36	42,1	48	33,3	60	24,4	72	15,7		
13	66,4	25	52,3	37	41,2	49	32,5	61	23,7	73	15,1		
14	65,2	26	51,3	38	40,4	50	31,8	62	22,8	74	14,5		
15	64,0	27	50,3	39	39,5	51	31,0	63	22,0	75	13,8		
16	62,8	28	49,2	40	38,8	52	30,3	64	21,2	76	13,0		
17	61,6	29	48,2	41	38,2	53	29,6	65	20,5	77	11,9		
18	60,3	30	47,3	42	37,5	54	28,9	66	19,7	78	10,9		
19	59,0	31	46,5	43	36,9	55	28,1	67	19,0	79	9,8		

Hence (if a table of the above kind be computed to correspond with every table of observations, that now are, or may be hereafter obtained) the values of annuities on single and combined lives, with their reversions, may be obtained nearly, by taking their complements, or the whole numbers nearest thereto, from such tables, and applying them to the proper solutions, instead of the differences between the given ages and 86.

EXAMPLE I.

If the values of the single lives of 10, and 70, at four per Cent. be required; according to the *London* observations.

The respective tabular complements of those ages are 70, and 17;

The ages, which (when the decrements of life are supposed to be equal) correspond to those complements are 16, and 69;

The values of which are (per table page 169, 171) 16, 3, and 6, 4:

And the values of these lives, according to Mr. *Simpson's* tables, are 16,4, and 6,5.

EXAMPLE II.

Let the value of the joint lives of the ages 10 and 31, at four *per Cent.* according to the *London* observations (which was found, by quest. 68, to be 10,8) be required by this method, applied to the approximation in quest. 64.

The value of the single life of 31 is, by Mr. *Simpson's* tables, 12,9; and the tabular complements of the given ages give 70, and 47.

Now if from 12,9 be taken $\left(\frac{n-1}{6r} = \frac{46}{6,24} =\right) 7,4,$

the remainder will be 5,5; this being multiplied by

$\left(\frac{n+1}{2m} = \frac{48}{140} =\right) \frac{12}{35},$ produces 1,9; which, taken

from 12,9, leaves 11,0, for the value required.

The method above given, for finding the complement of a life, according to any table of observations, may be extended to the finding the expectations of such lives.

The *expectation of life* is that time, which a person, of a given age, may justly expect to continue in being.

QUESTION CV.

What is the expectation of a single life of a given age?

By the process in quest. 104, when the decrements of life are equal, the sum of the probabilities, which a given life, whose complement is n , has of continuing to the extremity of old age, will be $\frac{n-1}{2}.$

This

This expression will not be the whole expectation of life ; because the series, $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} \&c.$

whose sum it is, consists (only) of the probabilities of the given life's surviving the first, second, third, $\&c.$ years, without any allowance for the time, which the life may continue after the end of such year, even though it should fail before the expiration of the next.

Now when a life has attained the beginning of that year, in which it may be supposed to fail ; it will be an equal chance, whether it fails in the first half year, or the latter half year thereof ; and therefore the probability of surviving half of the year should be added to each of the terms of that series.

Thus, in the first year, the probability of surviving to the end thereof is $\frac{n-1}{n}$, and that of failing $\frac{1}{n}$; therefore, if

$\frac{1}{2n}$, the half of the latter, be added to $\frac{n-1}{n}$, the for-

mer ; the sum $\left(\frac{n-1}{n} + \frac{1}{2n} = \right) \frac{2n-1}{2n}$, will be the ex-

pectation of life for that year ; and by arguing in the same manner, the expectation of life, for the second,

third, fourth, $\&c.$ years, will be $\frac{2n-3}{2n}, \frac{2n-5}{2n}, \frac{2n-7}{2n},$

$\&c.$ the n th term of which, *viz.* $\frac{2n-2n+1}{2n}$, will be

$\frac{1}{2n}$: This being an arithmetical progression, whose

greatest term is $\frac{2n-1}{2n}$; least term $\frac{1}{2n}$, and number of

terms n , therefore $\frac{2n-1}{2n} + \frac{1}{2n} \times \frac{n}{2}$, Or $\left(\frac{2n}{2n} \times \frac{n}{2} = \right) \frac{n}{2}$

will be the whole expectation of the given life.

Q 3

Hence

Hence the expectation of a single life, according to any table of observations, will be nearly equal to half the complement found by quest. 104.

QUESTION CVI.

What is the expectation of two joint lives, of given ages?

Let the complement of the elder life be n , and that of the younger m ; then, by arguing as before, the expectation of their joint continuance, for one, two, three, &c. years will be

$$\frac{2n-1}{2n} \times \frac{2m-1}{2m}, \frac{2n-3}{2n} \times \frac{2m-3}{2m}, \frac{2n-5}{2n} \times \frac{2m-5}{2m} \text{ \&c.}$$

and the sum of n terms of that series will be the expectation of the joint lives.

Now the terms of this series are the products of the two arithmetical progressions, $\frac{2n-1}{2n}, \frac{2n-3}{2n}, \frac{2n-5}{2n} (n)$,

and $\frac{2m-1}{2m}, \frac{2m-3}{2m}, \frac{2m-5}{2m} (n)$; whose common differ-

ences are $\frac{1}{n}$ and $\frac{1}{m}$; the sum of the first being $\frac{n}{2}$: And

since the greatest term of the second is $\frac{2m-1}{2m}$, the least

$\frac{2m-2n+1}{2m}$, and number of terms n ; the sum thereof will be

$$\left(\frac{2m-1}{2m} + \frac{2m-2n+1}{2m} \times \frac{n}{2} \right) \frac{4m-2n}{4m} \times n$$

$$\text{Or } 1 - \frac{n}{2m} \times n$$

Therefore

Therefore, by quest. 21, the sum of n terms of the series

of products will be $\frac{\frac{n}{2} \times 1 - \frac{n}{2m} \times n}{n} + \frac{\frac{n+1}{2} \times \frac{n-1}{2m-1}}{2 \cdot 2 \cdot 3} \times \left(\frac{1}{n} \times \frac{1}{m}\right);$

$$\text{Or } \frac{\frac{n}{2} \times 1 - \frac{n}{2m} \times n}{n} + \frac{\frac{n+1}{2} \times \frac{n-1}{2m-1}}{4 \cdot 3m};$$

$$\text{Or } \frac{n}{2} - \frac{nn}{4m} + \frac{nn-1}{12m}$$

Now, because $\frac{nn-1}{12m}$ differs from $\frac{nn}{12m}$ only by $\frac{1}{12m}$ (a quantity too small to affect this calculation) therefore the above may be wrote

$\left(\frac{n}{2} - \frac{nn}{4m} + \frac{nn}{12m}\right) = \frac{n}{2} - \frac{nn}{6m}$; which is the expectation of the two joint lives.

COROLLARY

Hence also, if the two ages be equal, the expectation of their joint lives will be $\left(\frac{n}{2} - \frac{nn}{6n} = \frac{n}{2} - \frac{n}{6} = \right) \frac{n}{3}$.

QUESTION CVII.

What is the expectation of three joint lives, of given ages?

Let the complements of the eldest, second, and youngest, be severally denoted by n , m , and t ; then will the expectation required (by arguing as before) be

$$\frac{2n-1}{2n} \times \frac{2m-1}{2m} \times \frac{2t-1}{2t} + \frac{2n-3}{2n} \times \frac{2m-3}{2m} \times \frac{2t-3}{2t} (n):$$

Now since the terms of this series are the products of the terms of three arithmetical progressions, whose sums

are $\frac{n}{2}$, $1 - \frac{n}{2m} \times n$, and $1 - \frac{n}{2t} \times n$; their common dif-

ferences

ferences $\frac{1}{n}, \frac{1}{m}, \frac{1}{t}$; and the number of their terms n ; therefore the sum of this series of products will (by quest. 22) be

$$\begin{aligned} & \frac{n}{2} \times 1 - \frac{n}{2m} \times 1 - \frac{n}{2t} + \frac{n+1 \cdot n \cdot n - 1}{2 \cdot 2 \cdot 3} \\ & \times \frac{2n-1}{2 \cdot n \cdot mt} + \frac{2m-1}{2m \cdot nt} + \frac{2t-1}{2t \cdot nm} - \frac{n+1 \cdot n \cdot n - 1^2}{2 \cdot 2 \cdot 2} \times \frac{1}{nmt}; \\ \text{Or } & \frac{n}{2} \times 1 - \frac{n}{2m} \times 1 - \frac{n}{2t} + \frac{n+1 \times n-1 \times 2n+2m+2t-3}{2 \cdot 2 \cdot 2 \cdot 3 \cdot mt} \\ & \left(- \frac{n+1 \cdot n - 1^2}{2 \cdot 2 \cdot 2 \cdot mt} \right); \\ \text{Or } & \frac{n}{2} \times \frac{2m-n}{2m} \times \frac{2t-n}{2t} + \frac{2n+2m+2t-3}{3} - n-1 \\ & \left(\times \frac{n+1 \cdot n - 1}{2 \cdot 2 \cdot 2 \cdot mt} \right); \end{aligned}$$

Which, being reduced, and nn wrote for $n+1 \times n-1$, as in the last question, will give

$$\begin{aligned} & \frac{12nmt - 4n^2t - 4n^2m + 2n^3}{24mt}, \text{ Or } \\ & \frac{n}{2} - \frac{n^2}{6} \times \frac{1}{m} + \frac{1}{t} + \frac{n^3}{12mt} \text{ for the expectation of three joint lives.} \end{aligned}$$

COROL. I.

Hence, if the lives be of equal ages, their expectation will be $\left(\frac{n}{2} - \frac{n^2}{6n} - \frac{n^2}{6n} + \frac{n^3}{12n^2} = \frac{n}{2} - \frac{2n}{6} + \frac{n}{12} = \frac{6n-4n+n}{12} = \right) \frac{n}{4}$

COROL. II.

Hence also the manner of finding the expectation of any number of joint lives is sufficiently evident, and may be expressed in the words used by Mr. De Moivre, viz.

The

The rule to find the expectation of any number of joint lives.

- " Take $\frac{1}{2}$ of the shortest complement ;
 " Take $\frac{1}{6}$ part of the square of the shortest, which divide,
 " successively, by all the other complements ; then add all
 " the quotients together :
 " Take $\frac{1}{24}$ part of the cube of the shortest complement ;
 " which divide, successively, by the products of all the other
 " complements, taken two and two :
 " Take $\frac{1}{120}$ part of the biquadrate of the shortest comple-
 " ment ; which divide, successively, by the products of all
 " the other complements, taken three and three, and so on.
 " Then from the result of the first operation, subtract the
 " result of the second, to the remainder add the result of the
 " third, from the sum subtract the result of the fourth, and
 " so on.
 " The last quantity, remaining after these alternate sub-
 " tractions and additions, will be the thing required.
 " N. B. The divisors 2, 6, 12, 20, &c. are the pro-
 " ducts of 1 by 2, of 2 by 3, of 3 by 4, of 4 by 5, &c.

COROL.

- " But if all the lives be equal, add unity to the number of
 " lives, and divide their complement by that number, thus
 " increased by unity ; and the quotient will express the time
 " due to their joint continuance."

QUESTION CVIII.

What is the expectation of the longest of two lives, of given ages ?

From the sum of the expectations of the single } $\frac{m}{2} + \frac{n}{2}$:
 lives

Take the expectation of the two joint lives - $\frac{m}{2} - \frac{nm}{6m}$:

The remainder will be the expectation of the } $\frac{m}{2} + \frac{nm}{6m}$:
 longest of those lives

COROL.

Hence, if the lives be of equal ages, the expectation of the longest will be $\left(\frac{n}{2} + \frac{nn}{6n} + \frac{n}{6} = \frac{2n}{3}\right)$.

QUESTION CIX.

What is the expectation of the longest of three lives, of given ages?

To the sum of the expectations of
the three single lives $\left\{ \frac{n}{2} + \frac{m}{2} + \frac{t}{2} \right\}$
Add the expectation of the three
joint lives $\left\{ \frac{n^2}{2} - \frac{n^2}{6m} - \frac{n^2}{6t} + \frac{n^3}{12mt} \right\}$

The sum will be $n + \frac{m}{2} + \frac{t}{2} - \frac{n^2}{6m} - \frac{n^2}{6t} + \frac{n^3}{12mt}$

To the expectation of the two joint lives, $\left\{ \frac{n}{2} - \frac{nn}{6m} \right\}$
whose complements are n and m

Add, that of the lives, whose complements are n and t , $\left\{ \frac{n}{2} - \frac{nn}{6t} \right\}$

And, that of the lives, whose complements are m and t , $\left\{ \frac{m}{2} - \frac{mm}{6t} \right\}$

The sum will be $n + \frac{m}{2} - \frac{nn}{6m} - \frac{nn}{6t} - \frac{mm}{6t}$

And the difference of these sums, $\frac{t}{2} + \frac{mm}{6t} + \frac{n^3}{12mt}$,
will be the expectation required.

COROLL. I.

If the lives be of equal ages, the expectation of the longest will be

$$\left(\frac{n}{2} + \frac{nn}{6n} + \frac{n^3}{12nn} = \frac{n}{2} + \frac{n}{6} + \frac{n}{12} = \frac{9n}{12} = \frac{3n}{4}\right)$$

COROLL. II.

Hence the method of finding the expectation of the longest of any number of lives may be expressed in words at length as follows: The

The rule for finding the expectation of the longest of any number of lives, of given ages.

Let the complement of the longest life be called the first complement; that of the next elder, the second; that of the next elder, the third; and so on.

Take half the first complement; divide one sixth part of the square of the second complement by the first complement; divide one twelfth part of the cube of the third complement by the products of the first and second; divide one twentieth part of the fourth power of the fourth complement by the continual product of the three former complements; and so continue, dividing the one thirtieth part of the fifth power of the fifth complement, the one forty-second part of the sixth power of the sixth complement, &c. by the continual product of all the former complements; so shall the sum of all these quotients be the expectation required.

But, if the lives are of equal ages, let their common complement be multiplied by the number of lives, and the product be divided by the number of lives more one; so shall the quotient be the expectation required.

If the expectation of any number of lives, according to a table of observations deduced from the bills of mortality of any place, be required; let the complements of those lives be found by question 104; and then proceed as directed in the above-given rules.

If any number is wanted that is beyond the reach of the succeeding table; the same may be found, by adding two, or more, of those together; for instance, the product of $\frac{1}{6r}$, by 65, will be the sum of the products thereof by 30 and 35, viz. (at four per Cent.) $4,8077\frac{1}{2} + 5,6089 = 10,3166$.

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TABLE the last, being the multiples of $\frac{1}{6r}$ very useful in computing the approximations to the values of combined lives.

years	3 per C.	$3\frac{1}{2}$ per C.	4 per C.	$4\frac{1}{2}$ per C.	5 per C.	6 per C.
1	0,1618	0,1610	0,1603	0,1595	0,1587	0,1572
2	0,3236	0,3221	0,3205	0,3190	0,3175	0,3145
3	0,4854	0,4831	0,4808	0,4785	0,4762	0,4717
4	0,6472	0,6441	0,6410	0,6380	0,6349	0,6289
5	0,8091	0,8052	0,8013	0,7975	0,7936	0,7862
6	0,9709	0,9662	0,9615	0,9569	0,9524	0,9434
7	1,1327	1,1272	1,1218	1,1164	1,1111	1,1006
8	1,2945	1,2882	1,2820	1,2759	1,2698	1,2579
9	1,4563	1,4493	1,4423	1,4354	1,4286	1,4151
10	1,6181	1,6103	1,6026	1,5949	1,5873	1,5723
11	1,7799	1,7713	1,7628	1,7544	1,7460	1,7296
12	1,9417	1,9324	1,9231	1,9139	1,9048	1,8868
13	2,1036	2,0934	2,0833	2,0734	2,0635	2,0440
14	2,2654	2,2544	2,2436	2,2329	2,2222	2,2013
15	2,4272	2,4155	2,4038	2,3924	2,3809	2,3585
16	2,5890	2,5765	2,5641	2,5518	2,5397	2,5157
17	2,7508	2,7375	2,7244	2,7113	2,6984	2,6730
18	2,9126	2,8986	2,8846	2,8708	2,8571	2,8302
19	3,0744	3,0596	3,0449	3,0303	3,0159	2,9874
20	3,2362	3,2206	3,2051	3,1898	3,1746	3,1447
21	3,3981	3,3817	3,3654	3,3493	3,3333	3,3019
22	3,5599	3,5427	3,5256	3,5088	3,4921	3,4591
23	3,7217	3,7037	3,6859	3,6683	3,6508	3,6164
24	3,8835	3,8647	3,8461	3,8278	3,8095	3,7736
25	4,0453	4,0258	4,0064	3,9872	3,9682	3,9308
26	4,2071	4,1868	4,1667	4,1467	4,1270	4,0881
27	4,3689	4,3478	4,3269	4,3062	4,2857	4,2453
28	4,5307	4,5089	4,4872	4,4657	4,4444	4,4025
29	4,6925	4,6699	4,6474	4,6252	4,6032	4,5598
30	4,8544	4,8305	4,8077	4,7847	4,7619	4,7170
31	5,0162	4,9920	4,9679	4,9442	4,9206	4,8742
32	5,1780	5,1530	5,1282	5,1037	5,0794	5,0315
33	5,3398	5,3140	5,2884	5,2632	5,2381	5,1887
34	5,5016	5,4751	5,4487	5,4227	5,3968	5,3459
35	5,6634	5,6361	5,6089	5,5821	5,5555	5,5032

F I N I S.